

Formality theorem with coefficients in a module

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Abstract

In this article, X will denote a \mathcal{C}^∞ manifold. In a very famous article, Kontsevich ([Ko]) showed that the differential graded Lie algebra (DGLA) of polydifferential operators on X is formal. Calaque ([C1]) extended this theorem to any Lie algebroid. More precisely, given any Lie algebroid E over X , he defined the DGLA of E -polydifferential operators, $\Gamma(X, {}^E D_{poly}^*)$, and showed that it is formal. Denote by $\Gamma(X, {}^E T_{poly}^*)$ the DGLA of E -polyvector fields. Considering M , a module over E , we define $\Gamma(X, {}^E T_{poly}^*(M))$ the $\Gamma(X, {}^E T_{poly}^*)$ -module of E -polyvector fields with values in M . Similarly, we define the $\Gamma(X, {}^E D_{poly}^*)$ -module of E -polydifferential operators with values in M , $\Gamma(X, {}^E D_{poly}^*(M))$. We show that there is a quasi-isomorphism of L_∞ -modules over $\Gamma(X, {}^E T_{poly}^*)$ from $\Gamma(X, {}^E T_{poly}^*(M))$ to $\Gamma(X, {}^E D_{poly}^*(M))$. Our result extends Calaque's (and Kontsevich's) result.

1 Introduction

In this article, X will denote a \mathcal{C}^∞ -manifold and \mathcal{O}_X will denote the sheaf of \mathcal{C}^∞ functions. To X are associated two sheaves of differential graded Lie algebras (DGLAs) T_{poly}^* and D_{poly}^* . The first one, T_{poly}^* is the sheaf of DGLAs of polyvector fields on X with differential zero and Schouten bracket. The second one, D_{poly}^* , is the sheaf of DGLAs of polydifferential operators on X with Hochschild differential and Gerstenhaber bracket. Kontsevich showed that there is a quasi-isomorphism of L_∞ -algebras from $\Gamma(X, T_{poly}^*)$ to $\Gamma(X, D_{poly}^*)$, that is to say that $\Gamma(X, D_{poly}^*)$ is formal. The aim of this article is to introduce a module in Kontsevich formality theorem.

Let us now consider a \mathcal{D}_X -module M . Inspired by the expression of the Schouten bracket, we endow $T_{poly}^*(M) = T_{poly}^* \otimes_{\mathcal{O}_X} M$ with a T_{poly}^* -module structure. Similarly, we can endow $D_{poly}^*(M) = D_{poly}^* \otimes_{\mathcal{O}_X} M$ with a D_{poly}^* -module structure as follows : if $P \in D_{poly}^p$ and $Q \in D_{poly}^q(M)$,

$$P \cdot_G Q = P \bullet Q - (-1)^{pq} Q \bullet P$$

with

$$\forall a_0, \dots, a_{p+q} \in \mathcal{O}_X, \\ (P \bullet Q)(a_0, \dots, a_{p+q}) = \sum_{i=0}^p (-1)^{iq} P(a_0, \dots, a_{i-1}, Q(a_i, \dots, a_{i+q}), \dots, a_{p+q}).$$

The formula makes sense because Q is a differential operator with coefficients in a \mathcal{D}_X -module M . The expression $Q \bullet P$ is defined in an analogous way. The differential on $D_{poly}^*(M)$ is given by the action of the multiplication μ , $\mu \cdot_G -$. Using Kontsevich's formality theorem, one may see $D_{poly}^*(M)$ as an L_∞ -module over T_{poly}^* and we will prove that it is formal. We will work in the more general setting of Lie algebroids.

Let us now consider a Lie algebroid E . To E is associated a sheaf of E -differential operators, $D(E)$ ([R]). Lie algebroids generalize at the same time the sheaf of vector fields on a manifold (in this case $E = TX$ and $D(E) = \mathcal{D}_X$) and Lie algebras (in this case $D(E)$ is the enveloping algebra). Lie algebroids have been extensively studied recently because many examples of Lie algebroids arise from geometry (Poisson manifolds, group actions, foliations ...). To E , one can associate the sheaf of DGLAs of E -polyvectorfields ${}^E T_{poly}^* = \bigoplus_{k=-1}^{\infty} \wedge^{k+1} E$ with zero differential and a Schouten type Lie bracket ([C1]). Calaque has given an appropriate generalization of the notion of polydifferential operators. In [C1], he defines the DGLA of E -polydifferential operators, $\Gamma(X, {}^E D_{poly}^*)$, and constructs an L_∞ quasi-isomorphism from $\Gamma(X, {}^E T_{poly}^*)$ to $\Gamma(X, {}^E D_{poly}^*)$.

Let us now consider a $D(E)$ -module M . We can perform the construction described above and define the ${}^E T_{poly}^*$ -module ${}^E T_{poly}^*(M)$ (the sheaf of the E -polyvectors with coefficients in M) and the ${}^E D_{poly}^*$ -module ${}^E D_{poly}^*(M)$ (the sheaf of E -polydifferential operators with coefficients in M). By Calaque's result $\Gamma(X, {}^E D_{poly}^*(M))$ is an L_∞ -module over $\Gamma(X, {}^E T_{poly}^*)$. The main result of the paper is the following theorem.

Theorem 3.4.1 :

There is a quasi-isomorphism of L_∞ -modules over $\Gamma(X, {}^E T_{poly})$ from $\Gamma(X, {}^E T_{poly}(M))$ to $\Gamma(X, {}^E D_{poly}(M))$.

Our result extends Calaque's formality theorem ([C1], take $M = \mathcal{O}_X$) and Kontsevich's formality theorem ([Ko], take $M = \mathcal{O}_X$ and $E = TX$).

If X is a Poisson manifold, we know from Kontsevich's work ([Ko]) that there is a star product on $\mathcal{O} = \Gamma(\mathcal{O}_X)$. Let \mathcal{M} be a \mathcal{D}_X -module and $M = \Gamma(X, \mathcal{M})$. Using the star product, we can endow $M[[h]]$ with a $\mathcal{O}[[h]] \otimes \mathcal{O}[[h]]^{op}$ -module structure. If $\pi \in \Gamma(\wedge^2 TX)$ is the bivector defining the Poisson structure on \mathcal{O} , $h\pi$ defines a Poisson structure on the algebra $\mathcal{O}[[h]]$. As a corollary of our theorem, we get an isomorphism from the Poisson cohomology of the Poisson algebra $\mathcal{O}[[h]]$ with coefficients in $M[[h]]$ and the differential Hochschild cohomology of $\mathcal{O}[[h]]$ with coefficients in $M[[h]]$.

Our proofs are analogous to that of [D1], [C1], [D2], [CDH]. We use Kontsevich's formality theorem for \mathbb{R}_{formal}^d and a Fedosov like globalization techniques.

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Notation :

For a study of L_∞ structures, we refer to [AMM], [D2], [D3], [HS], [LS].

Let k be a field of characteristic zero and let V be a \mathbb{Z} -graded k -vector space

$$V = \bigoplus_{i \in \mathbb{Z}} V_i.$$

If x is in V_i , we set $|x| = i$. We will always assume that the gradation is bounded below. Recall the definition of the graded symmetric algebra and

the graded wedge algebra :

$$\begin{aligned} S(V) &= \frac{T(V)}{\langle x \otimes y - (-1)^{|x||y|} y \otimes x \rangle} \\ \wedge(V) &= \frac{T(V)}{\langle x \otimes y + (-1)^{|x||y|} y \otimes x \rangle}. \end{aligned}$$

If i is in \mathbb{Z} , we will denote by $V[i]$ the graded vector space defined by $V[i]^n = V^{i+n}$.

Denote by $S^c(V)$ the cofree cocommutative coalgebra without counity cofreely cogenerated by V . As a vector space $S^c(V)$ is $S^+(V)$. Its comultiplication is given by

$$\Delta(x_1 \dots x_n) = \sum_{\substack{I \sqcup J = [1, n] \\ I \neq \emptyset \\ J \neq \emptyset}} (-1)^{\epsilon(I, J)} x_I \otimes x_J.$$

where $\epsilon(I, J)$ is the number of inversion of odd elements when going from $x_I x_J$ to $x_1 \dots x_n$. A coderivation Q on $S^c(V)$ is determined by its Taylor coefficients $Q^{[n]} : S^n(V) \rightarrow V$ (obtained by composing Q with the projection from $S(V)$ onto V).

An L_∞ algebra is a couple (L, Q) where L is a graded vector space and Q is a degree 1 two-nilpotent coderivation of $S^c(L[1]) = C(L)$. The coderivation Q is determined by its Taylor coefficients $(Q^{[n]})_{n \geq 1}$. Using an isomorphism between $S^n(L[1])$ and $\wedge^n(L)[n]$, the Taylor coefficients may be seen as maps

$\overline{Q}^{[n]} : \wedge^n L \rightarrow L[2-n]$. A differential graded Lie algebra $(L, d, [,])$ (with differential d and Lie bracket $[,]$) gives rise to an L_∞ -algebra determined by $\overline{Q}^{[1]} = d$, $\overline{Q}^{[2]} = [,]$ and $\overline{Q}^{[i]} = 0$ for $i \geq 2$.

Let L be a differential graded Lie algebra. We will say that it is a filtered DGLA if it is equipped with a complete descending filtration $\dots \mathcal{F}^1 L \subset \mathcal{F}^0 L = L$ such that $L = \lim_n L/\mathcal{F}^n L$. A Maurer Cartan element of L is an element x of $\mathcal{F}^1 L^1$ such that $Q^{[1]}x + \frac{1}{2}Q^{[2]}(x^2) = 0$.

Let (L_1, Q_1) and (L_2, Q_2) be two L_∞ -algebras. An L_∞ -morphism F from (L_1, Q_1) to (L_2, Q_2) is a morphism of coalgebras $F : C(L_1) \rightarrow C(L_2)$ compatible with coderivations (this means that $F \circ Q_1 = Q_2 \circ F$). As F is a morphism of coalgebras, it is determined by its Taylor coefficients

$\left(F^{[n]} : S^n(L_1[1]) \rightarrow L_2[1]\right)_{n \geq 1}$ or $\left(\overline{F}^{[n]} : \wedge^n(L_1) \rightarrow L_2[1-n]\right)_{n \geq 1}$. The relation $F \circ Q_1 = Q_2 \circ F$ boils down to say that $F^{[n]}$ satisfy an infinite collection of equations.

Let (L_1, Q_1) and (L_2, Q_2) be two filtered DGLAs and let F be an L_∞ -morphism from (L_1, Q_1) to (L_2, Q_2) compatible with these filtrations. If x is a Maurer Cartan element of L_1 , then $\sum_{n \geq 1} \frac{F^{[n]}(x^n)}{n!}$ is a Maurer Cartan element of L_2 .

Let L be an L_∞ -algebra and M a graded vector space. We will consider the $C(L)$ -comodule $S(L[1]) \otimes M$ with the coaction

$$\mathbf{a}(x_1 \dots x_n \otimes v) = \sum_{\substack{I \sqcup J = [1, n] \\ I \neq \emptyset}} (-1)^{\epsilon(I, J)} x_I \otimes (x_J \otimes v)$$

where $\epsilon(I, J)$ is the number of inversion of odd elements when going from $x_I x_J$ to $x_1 \dots x_n$. An L_∞ -module is a couple (M, ϕ) where ϕ is a degree 1 two-nilpotent coderivation of the $C(L)$ -comodule $S(L[1]) \otimes M$. The coderivation ϕ is determined by its Taylor coefficients $\phi^{[n]} : S^n(L[1]) \otimes M \rightarrow M[1]$ or $\overline{\phi}^{[n]} : \wedge^n(L) \otimes M \rightarrow M[1-n]$. The map $\phi^{[0]}$ is a differential on M . A module M over a differential graded Lie algebra $(L, d, [,])$ is an L_∞ -module with Taylor coefficients $\overline{\phi}^{[0]} = d$, $\overline{\phi}^{[1]}(X \otimes m) = X \cdot m$ ($X \in L, m \in M$) and $\overline{\phi}^{[n]} = 0$ if $n > 1$.

Let (M_1, ϕ_1) and (M_2, ϕ_2) be two L_∞ -modules. An L_∞ -morphism \mathcal{V} from (M_1, ϕ_1) to (M_2, ϕ_2) is a (degree 0) morphism of comodules from $S(L[1]) \otimes M_1$ to $S(L[1]) \otimes M_2$ such that $\mathcal{V} \circ \phi_1 = \phi_2 \circ \mathcal{V}$. It is determined by its Taylor coefficients $\left(\mathcal{V}^{[n]} : S^n(L[1]) \otimes M_1 \rightarrow M_2\right)_{n \geq 0}$ or $\left(\overline{\mathcal{V}}^{[n]} : \wedge^n(L) \otimes M_1 \rightarrow M_2[-n]\right)_{n \geq 0}$. The compatibility of \mathcal{V} with coderivation is expressed by an infinite collection of equations satisfied by $\mathcal{V}^{[n]}$.

In this text, DGLA (repectively DGAA) will stand for differential graded Lie algebra (respectively differential graded associative algebra).

We assume Einstein convention for the summation over repeated indices.

If \mathcal{F} is a sheaf over X , then $\Gamma(\mathcal{F})$ denotes its global sections. If \mathcal{F} and \mathcal{G} are two sheaves and if $\Theta : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves, then $\Theta(X)$ will denote the morphism from $\Gamma(\mathcal{F})$ to $\Gamma(\mathcal{G})$ contained in Θ

2 Recollections

2.1 Lie algebroids : definitions and first properties

Let X be a \mathcal{C}^∞ -manifold and let \mathcal{O}_X be the sheaf of \mathcal{C}^∞ functions on X . Let Θ_X be the \mathcal{O}_X -module of \mathcal{C}^∞ vector fields on X .

Definition 2.1.1 *A sheaf in \mathbb{R} -Lie algebras over X , E , is a sheaf of \mathbb{R} -vector spaces such that for any open subset U , $E(U)$ is equipped with the structure of a Lie algebra and the restriction morphisms are Lie algebra homomorphisms.*

A morphism between two sheaves of Lie algebras E and F is a \mathbb{R}_X -module morphism which is a Lie algebra morphism on each open subset.

Definition 2.1.2 *A Lie algebroid over X is a pair (E, ω) where*

- *E is a locally free \mathcal{O}_X -module of finite constant rank that is to say a vector bundle over X ,*
- *E is a sheaf of \mathbb{R} -Lie algebras,*
- *$\omega : E \rightarrow \Theta_X$ is an \mathcal{O}_X -linear morphism of sheaves of \mathbb{R} -Lie algebras such that the following compatibility relation holds :*

$$\forall (\xi, \zeta) \in E^2, \quad \forall f \in \mathcal{O}_X, \quad [\xi, f\zeta] = \omega(\xi)(f)\zeta + f[\xi, \zeta].$$

One calls ω the anchor map. When there is no ambiguity, we will drop the anchor map in the notation of the Lie algebroid.

For example TX is a Lie algebroid on X and a finite dimensional Lie algebra is a Lie algebroid over a point. Other examples arises from Poisson manifolds, foliations, Lie group actions (see [F] for example).

A Lie algebroid (E, ω) gives rise to the sheaf of E -differential operators generated by \mathcal{O}_X and E which is denoted by $D(E)$.

Definition 2.1.3 *$D(E)$ is the sheaf associated to the presheaf:*

$$U \mapsto T_{\mathbb{R}}^+(\mathcal{O}_X(U) \oplus E(U)) / J_U$$

where J_U is the two sided ideal generated by the relations

$$\forall (f, g) \in \mathcal{O}_X(U), \quad \forall (\xi, \zeta) \in E(U)^2$$

$$\begin{aligned} 1) & f \otimes g = fg \\ 2) & f \otimes \xi = f\xi \\ 3) & \xi \otimes \zeta - \zeta \otimes \xi = [\xi, \zeta] \\ 4) & \xi \otimes f - f \otimes \xi = \omega(\xi)(f) \end{aligned}$$

If $E = TX$, $D(E)$ is the sheaf of differential operators on X , \mathcal{D}_X . If E is a finite dimensional Lie algebra \mathfrak{g} , $D(E)$ is $U(\mathfrak{g})$, the enveloping algebra of \mathfrak{g} .

$D(E)$ is also endowed with a coassociative \mathcal{O}_X -linear coproduct $\Delta : D(E) \rightarrow D(E) \otimes_{\mathcal{O}_X} D(E)$ defined as follows (see [X] example 3.1):

$$\begin{aligned} \Delta(1) &= 1 \otimes 1 \\ \forall u \in E, \quad \Delta(u) &= u \otimes 1 + 1 \otimes u \\ \forall (P, Q) \in D(E)^2, \quad \Delta(PQ) &= \Delta(P)\Delta(Q). \end{aligned}$$

Let M be a $D(E)$ -module. The cohomology of E with coefficients in M , is computed by the complex $(Hom_{\mathcal{O}_X}(\wedge^* E, M), {}^E d_M)$ where ${}^E d_M$ is given by

$$\forall \phi \in Hom_{\mathcal{O}_X}(\wedge^n E, M), \forall u_0, \dots, u_n \in E,$$

$$\begin{aligned} {}^E d_M \phi(u_0, \dots, u_n) &= \sum_{i=1}^n (-1)^i u_i \cdot \phi(u_1, \dots, \widehat{u_i}, \dots, u_n) \\ &+ \sum_{i < j} (-1)^{i+j} \phi([u_i, u_j], u_0, \dots, \widehat{u_i}, \dots, \widehat{u_j}, \dots, u_n) \end{aligned}$$

Recall that \mathcal{O}_X has a natural left $D(E)$ -module structure defined by :

$$\forall f \in \mathcal{O}_X, \forall P \in D(E), \quad P \cdot f = \omega(P)(f).$$

If $M = \mathcal{O}_X$, we set ${}^E d_M = {}^E d$ and the complex above will be called the Lie cohomology complex of E .

If M is a $D(E)$ -module, a tensor with coefficients in M is a section of $M \otimes (\otimes E^*) \otimes (\otimes E)$.

The notion of connections has been extended to Lie algebroids (see [F] for example). Let \mathcal{B} be an \mathcal{O}_X -module. A E -connection on \mathcal{B} is a linear operator

$$\nabla : \Gamma(\mathcal{B}) \rightarrow {}^E\Omega^1(\Gamma(\mathcal{B})) = \Gamma\left(\text{Hom}_{\mathcal{O}_X}(\wedge^1 E, \mathcal{B})\right)$$

satisfying the following equation : for any $f \in \Gamma(\mathcal{O}_X)$ and any $v \in \Gamma(\mathcal{B})$

$$\nabla(fv) = {}^E d(f)v + f\nabla(v).$$

If u is an element of E , the connection ∇ defines a map $\nabla_u : \mathcal{B} \rightarrow \mathcal{B}$.

Assume now that \mathcal{B} is a bundle. If (e_1, \dots, e_d) is a local basis of E and (b_1, \dots, b_n) is a local basis of \mathcal{B} , one has

$$\nabla_{e_i}(b_j) = \Gamma_{i,j}^k b_k.$$

The connection ∇ is determined by its Christoffel symbol $\Gamma_{i,j}^k$.

Definition 2.1.4 *The curvature R of a connection ∇ with values in \mathcal{B} is the section R of the bundle $E^* \otimes E^* \otimes \mathcal{B}^* \otimes \mathcal{B}$ defined by : for any u, v in $\Gamma(E)$ and b in $\Gamma(\mathcal{B})$*

$$R(u, v)(b) = \left(\nabla_u \circ \nabla_v - \nabla_v \circ \nabla_u - \nabla_{[u, v]} \right) (b)$$

The curvature tensor is locally determined by the $(R_{i,j})_k^l$ defined by

$$R(e_i, e_j)b_k = (R_{i,j})_k^l b_l.$$

For a connection ∇ on $\mathcal{B} = E$, one can define the torsion tensor.

Definition 2.1.5 *The torsion of ∇ is a section of $E \otimes E^* \otimes E^*$ defined by : for any u, v in $\Gamma(E)$,*

$$T(u, v) = \nabla_u(v) - \nabla_v(u) - [u, v].$$

Proposition 2.1.6 *A torsion free connection on E exists.*

A proof of this proposition can be found in [C2].

Examples of $D(E)$ -modules

Example 1 :

Flat connections provides examples of $D(E)$ -modules.

Example 2 :

If E is a Lie algebroid with anchor map ω , then $\text{Ker}\omega$ is a left $D(E)$ -module for the following operations : for all f in \mathcal{O}_X , for all ξ in E , for all σ in $\text{Ker}\omega$,

$$f \cdot \sigma = f\sigma, \quad \xi \cdot \sigma = [\xi, \sigma].$$

Example 3 :

If M and N are two left $D(E)$ -modules, then (see [Bo] for the \mathcal{D}_X -module case and [Ch2]) $M \otimes_{\mathcal{O}_X} N$ and $\mathcal{H}om_{\mathcal{O}_X}(M, N)$, endowed with the two operations described below, are left $D(E)$ -modules :

$$\begin{aligned} \forall m \in M, \forall n \in N, \forall a \in \mathcal{O}_X, \forall \xi \in E, \\ a \cdot (m \otimes n) \cdot a = a \cdot m \otimes n \\ \xi \cdot (m \otimes n) = \xi \cdot m \otimes n + m \otimes \xi \cdot n \end{aligned}$$

$$\begin{aligned} \forall \phi \in \mathcal{H}om_{\mathcal{O}_X}(M, N), \forall m \in M, \forall a \in \mathcal{O}_X, \forall \xi \in E, \\ (a \cdot \phi)(m) = a\phi(m) \\ (\xi \cdot \phi)(m) = \xi \cdot \phi(m) - \phi(\xi \cdot m). \end{aligned}$$

Example 4 :

It is a well known fact ([Bo], [Ka]) that the \mathcal{O}_X -module of differential forms of maximal degree, $\Omega_X^{\dim X}$, is endowed with a right \mathcal{D}_X -module structure. We may extend this result ([Ch1]) to $\Lambda^d(E^*)$ where d be the rank of E . Indeed E acts on $\Lambda^d(E^*)$ by the adjoint action. The action of an element ξ of E is called the Lie derivative of ξ and is denoted L_ξ . The \mathcal{O}_X -module $\Lambda^d(E^*)$, endowed with the following operations,

$$\begin{aligned} \forall \sigma \in \Lambda^d(E^*), \forall \xi \in E, \forall f \in \mathcal{O}_X, \\ \sigma \cdot a = a\sigma \\ \sigma \cdot \xi = -L_\xi(\sigma) \end{aligned}$$

is a right $D(E)$ -module.

Example 5 :

If \mathcal{M} and \mathcal{N} are two right $D(E)$ -modules, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$, endowed with the two following operations,

$$\begin{aligned} \forall \phi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}), \forall m \in \mathcal{M}, \forall a \in \mathcal{O}_X, \forall \xi \in E, \\ (a \cdot \phi)(m) = \phi(m) \cdot a \\ (\xi \cdot \phi)(m) = -\phi(m) \cdot \xi + \phi(m \cdot \xi) \end{aligned}$$

is a left $D(E)$ -module ([Ch2]). This was already known for D -modules. In particular $\mathcal{H}om_{\mathcal{O}_X}(\Lambda^d(E^*), \Omega_X^{dim X})$ is a left $D(E)$ -module which is used in [ELW] to define the modular class of E .

Example 6 :

If \mathcal{M} is a right $D(E)$ -module and \mathcal{N} is a left $D(E)$ -module, then $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$, endowed with the two following operations,

$$\begin{aligned} \forall m \in \mathcal{M}, \forall n \in \mathcal{N}, \forall a \in \mathcal{O}_X, \forall \xi \in E, \\ (m \otimes n) \cdot a = m \otimes a \cdot n = m \cdot a \otimes n \\ (m \otimes n) \cdot \xi = m \cdot \xi \otimes n - m \otimes \xi \cdot n \end{aligned}$$

is a right $D(E)$ -module (see [Bo] for D -modules and [Ch2]). Given any $D(E)$ -module which is locally free of rank one, the functor $\mathcal{N} \mapsto \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{N}$ establishes an equivalence of categories between left and right $D(E)$ -modules. Its inverse functor is given by $\mathcal{M} \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M})$. This equivalence of categories is well known for D -modules ([Bo], [Ka]) and was generalized to Lie algebroids in [Ch2]. In the case where $X = \mathbb{R}^d$ and $E = T\mathbb{R}^d$, this equivalence of categories is particularly simple because we may choose $dx^1 \wedge \dots \wedge dx^d$ as a basis of the $\mathcal{O}_{\mathbb{R}^d}$ -module Ω_X^d . There exists a unique anti-isomorphism of $\mathcal{D}_{\mathbb{R}^d}$, σ , such that $\sigma(f) = f$ and $\sigma(\frac{\partial}{\partial x^i}) = -\frac{\partial}{\partial x^i}$. Any left $\mathcal{D}_{\mathbb{R}^d}$ -module can be seen as a right $\mathcal{D}_{\mathbb{R}^d}$ -module (and conversely) in the following way :

$$\forall P \in \mathcal{D}_{\mathbb{R}^d}, \forall m \in M, \quad m \cdot P = \sigma(P) \cdot m.$$

Example 7 :

Let $\mathcal{D}b_X$ the sheaf of distributions over X . As \mathcal{O}_X is a left \mathcal{D}_X -module, $\mathcal{D}b_X$ is a right \mathcal{D}_X -module (by transposition).

Example 8 :

Let us recall our definition of a Lie algebroid morphism ([Ch2]) which coincides with that of Almeida and Kumpera ([AK])

Definition 2.1.7 *Let (E_X, ω_X) (respectively (E_Y, ω_Y)) be a Lie algebroid over X (respectively Y). A morphism Φ from (E_X, ω_X) to (E_Y, ω_Y) is a pair (f, F) such that*

- $f : X \rightarrow Y$ is a \mathcal{C}^∞ -morphism

- $F : E_X \rightarrow f^*E_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}E_Y$ such that the two following conditions are satisfied:

1) the diagram

$$\begin{array}{ccc} E_X & \xrightarrow{F} & f^*E_Y \\ \omega_X \downarrow & & \downarrow f^*\omega_Y \\ \Theta_X & \xrightarrow{Tf} & f^*\Theta_Y \end{array}$$

commutes.

2) Let ξ and η be two elements of E_X^2 . Put $F(\xi) = \sum_{i=1}^m a_i \otimes \xi_i$ and

$$F(\eta) = \sum_{j=1}^m b_j \otimes \eta_j, \text{ then}$$

$$F([\xi, \eta]) = \sum_{j=1}^n \omega_X(\xi)(b_j) \otimes \eta_j - \sum_{i=1}^n \omega_X(\eta)(a_i) \otimes \xi_i + \sum_{i,j} a_i b_j \otimes [\xi_i, \eta_j].$$

If $\Phi = (f, F)$ is Lie algebroid morphism from (E_X, ω_X) to (E_Y, ω_Y) and \mathcal{M} is a $D(E_Y)$ -module, then $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M}$ endowed with the two following operations

$$\forall (a, b) \in \mathcal{O}_X^2, \forall \xi \in E_X, \forall m \in f^{-1}\mathcal{M}$$

$$a \cdot (b \otimes m) = ab \otimes m$$

$$\xi \cdot (b \otimes m) = \omega_X(\xi)(b) \otimes m + \sum_i b a_i \otimes \xi_i m$$

(where $F(\xi) = \sum_i a_i \otimes \xi_i$ with a_i in \mathcal{O}_X and ξ_i in $f^{-1}E_Y$) is a left $D(E_X)$ -module ([Ch2]).

Morphisms of Lie algebroids generalize at the same time Lie algebra morphisms and morphisms between \mathcal{C}^∞ -manifolds. Examples of Lie algebroid morphisms can be found in [Ch3]. The $D(E_X) \otimes f^{-1}D(E_Y)^{op}$ -module $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D(E)$ generalizes the transfer module for D -modules (see [Bo], [Ka], [Ch2])

2.2 The sheaves of DGLAs ${}^E T_{poly}$ and ${}^E D_{poly}$

The sheaf of DGLAs of polyvectorfields can be extended to the Lie algebroids setting. The sheaf of DGLAs ${}^E T_{poly}$ of E -polyvector fields is defined as follows ([C1]):

$${}^E T_{poly} = \bigoplus_{k \geq -1} {}^E T_{poly}^k = \bigoplus_{k \geq -1} \wedge^{k+1} E$$

endowed with the zero differential and the Lie bracket $[,]_S$ uniquely defined by the two following properties :

$$\begin{aligned} & \bullet \forall f, g \in \mathcal{O}_X, [f, g]_S = 0 \\ & \bullet \forall \xi \in E, \forall f \in \mathcal{O}_X, [\xi, f]_S = \omega(\xi)(f) \\ & \bullet \forall \xi, \eta \in E, [\xi, \eta]_S = [\xi, \eta]_E \\ & \bullet \forall u \in {}^E T_{poly}^k, v \in {}^E T_{poly}^l, w \in {}^E T_{poly}, \\ & [u, v \wedge w]_S = [u, v]_S \wedge w + (-1)^{k(l+1)} v \wedge [u, w]_S \end{aligned}$$

In [C1], D. Calaque extended the sheaf of DGLAs of polydifferential operators to the Lie algebroid setting. Before recalling his construction, let us fix some notations.

Notation :

Let M_0, M_1, \dots, M_n be $n+1$ $D(E)$ -modules. Denote by $\pi_i : D(E) \rightarrow \text{End}(M_i)$ the maps defined by these actions. An element $P_0 \otimes_{\mathcal{O}_X} \dots \otimes_{\mathcal{O}_X} P_n$ of $D(E)^{\otimes n+1}$ defines a map

$$\begin{aligned} \pi_0(P_0) \otimes \dots \otimes \pi_{n+1}(P_{n+1}) : M_0 \otimes_{\mathbb{R}_X} \dots \otimes_{\mathbb{R}_X} M_n & \rightarrow M_0 \otimes_{\mathcal{O}_X} \dots \otimes_{\mathcal{O}_X} M_n \\ m_0 \otimes_{\mathbb{R}} \dots \otimes_{\mathbb{R}} m_n & \mapsto \pi_0(P_0)(m_0) \otimes_{\mathcal{O}_X} \dots \otimes_{\mathcal{O}_X} \pi_n(P_n)(m_n). \end{aligned}$$

In the sequel, we will be in the following situation $M_0, \dots, M_{i-1}, M_{i+1}, \dots, M_n$ are $D(E)$ endowed with left multiplication. If P is in $D(E)$, we will then write P for left multiplication with P which amounts to omit π_i . The $D(E)$ -module M_i will be \mathcal{O}_X (with its natural $D(E)$ -module structure) and we will write ω (as the anchor map) for the map from $D(E)$ to $\text{End}(\mathcal{O}_X)$.

Calaque defines the sheaf of DGLAs ${}^E D_{poly}^*$ of E -polydifferential operators as follows :

$${}^E D_{poly}^* = \bigoplus_{k \geq -1} {}^E D_{poly}^k$$

where

$$\begin{aligned} {}^E D_{poly}^{-1} &= \mathcal{O}_X \\ {}^E D_{poly}^k &= D(E)^{\otimes_{\mathcal{O}_X}^{k+1}} \text{ if } k \geq 0. \end{aligned}$$

Before defining the Lie bracket over ${}^E D_{poly}^*$, we need to introduce the bilinear product of degree 0,

$$\bullet : {}^E D_{poly}^* \otimes {}^E D_{poly}^* \rightarrow {}^E D_{poly}^*.$$

Let P (respectively Q) be an homogeneous element of ${}^E D_{poly}^*$ of positive degree $|P|$ (respectively $|Q|$), and let f (respectively g) be an element of ${}^E D_{poly}^{-1} = \mathcal{O}_X$. We have :

$$\begin{aligned} P \bullet Q &= \sum_{i=0}^{|P|} (-1)^{i|Q|} \left(id^{\otimes i} \otimes \Delta(|Q|) \otimes id^{\otimes |P|-i} \right) (P) \cdot (1^{\otimes i} \otimes_{\mathbb{R}} Q \otimes_{\mathbb{R}} 1^{\otimes |P|-i}) \\ P \bullet f &= \sum_{i=0}^{|P|} (-1)^i \left(id^{\otimes i} \otimes \omega \otimes id^{\otimes |P|-i} \right) (P) \cdot (1^{\otimes i} \otimes_{\mathbb{R}} f \otimes_{\mathbb{R}} 1^{\otimes |P|-i}) \\ f \bullet g &= 0 \\ f \bullet P &= 0 \end{aligned}$$

The Lie bracket between $P_1 \in {}^E D_{poly}^{k_1}$ and $P_2 \in {}^E D_{poly}^{k_2}$ is

$$[P_1, P_2] = P_1 \bullet P_2 - (-1)^{k_1 k_2} P_2 \bullet P_1.$$

The differential on ${}^E D_{poly}$ is $\partial = [1 \otimes 1, -]$.

Calaque has proved the following theorem ([C1]) which generalizes Kontsevitch's result ([Ko])

Theorem 2.2.1 *There exists a quasi-isomorphism of L_∞ -algebras, Υ , from $\Gamma({}^E T_{poly}^*)$ to $\Gamma({}^E D_{poly}^*)$. In other words, $\Gamma({}^E D_{poly}^*)$ is formal.*

3 Main results

Let E be a Lie algebroid over a manifold X and let $D(E)$ be the sheaf of E -differential operators. We will denote by M a left $D(E)$ -module.

3.1 The ${}^E T_{poly}^*$ -module ${}^E T_{poly}^*(M)$.

We introduce the complex ${}^E T_{poly}^*(M)$ of E -polyvector fields with values in M

$${}^E T_{poly}^*(M) = \bigoplus_{k \geq -1} {}^E T_{poly}^k(M) = \bigoplus_{k \geq -1} \wedge^{k+1} E \otimes M$$

with differential zero. If m is in M , we will identify m with $1 \otimes m$.

Proposition 3.1.1 ${}^E T_{poly}^*(M)$ is endowed with a ${}^E T_{poly}^*$ -module structure described as follows : for all $u = \xi_1 \wedge \dots \wedge \xi_{k+1} \in {}^E T_{poly}^k$, $v \in {}^E T_{poly}^l$ (with $k, l \geq 0$), $f \in \mathcal{O}_X$, $m \in M$,

$$\begin{aligned} \bullet f \cdot_S m &= 0 \\ \bullet (\xi_1 \wedge \dots \wedge \xi_{k+1}) \cdot_S m &= \sum_{i=1}^{k+1} (-1)^{k+1-i} \xi_1 \wedge \dots \wedge \widehat{\xi_i} \wedge \dots \wedge \xi_{k+1} \otimes \xi_i \cdot m \\ \bullet f \cdot_S (v \otimes m) &= [f, v]_S \otimes m \\ \bullet u \cdot_S (v \otimes m) &= [u, v]_S \otimes m + (-1)^{k(l+1)} v \wedge u \cdot_S m. \end{aligned}$$

When there is no ambiguity, we will drop the subscript S in the notation of the action of ${}^E T_{poly}^*$ over ${}^E T_{poly}^*(M)$.

Proof of the proposition

It is easy to check that the actions above are well defined. Let a be in ${}^E T_{poly}^s$. We need to verify that the following relation holds

$$u \cdot (v \cdot (a \otimes m)) - (-1)^{kl} v \cdot (u \cdot (a \otimes m)) = [u, v] \cdot (a \otimes m).$$

A straightforward computation shows that it is enough to check this relation for $a = 1$, which we will assume. We will need the two following lemmas.

Lemma 3.1.2 if $a \in {}^E T_{poly}^*$, $u \in {}^E T_{poly}^k$, $v \in {}^E T_{poly}^l$ ($k, l \geq -1$), one has

$$u \cdot (v \wedge a \otimes m) = [u, v] \wedge a \otimes m + (-1)^{k(l+1)} v \wedge u \cdot (a \otimes m).$$

Proof of the lemma : It is a straightforward computation. \square

Lemma 3.1.3 Let $a \in {}^E T_{poly}^*$, $m \in M$, $k, l \geq 0$, $u \in {}^E T_{poly}^k$, $v \in {}^E T_{poly}^l$. One has the following relation

$$(u \wedge v) \cdot (a \otimes m) = u \wedge (v \cdot (a \otimes m)) + (-1)^{(k+1)(l+1)} v \wedge (u \cdot (a \otimes m))$$

Proof of the lemma :

An easy computation shows that we may assume $a = 1$. The proof of the lemma goes by induction over k . The case $k = 0$ is obvious so that we assume $k \geq 1$. Set $u = \xi_1 \wedge \dots \wedge \xi_{k+1}$ and $u' = \xi_2 \wedge \dots \wedge \xi_{k+1}$ so that $u = \xi_1 \wedge u'$. Using the induction hypothesis and the case $k = 0$, we get the following sequence of equalities.

$$\begin{aligned} & (u \wedge v) \cdot m \\ &= (-1)^{l+k+1} (u' \wedge v) \otimes \xi_1 \cdot m + \xi_1 \wedge ((u' \wedge v) \cdot m) \\ &= (-1)^{l+k+1+k(l+1)} v \wedge u' \otimes \xi_1 \cdot m + \xi_1 \wedge u' \wedge (v \cdot m) + (-1)^{k(l+1)} \xi_1 \wedge v \wedge (u' \cdot m) \\ &= u \wedge (v \cdot m) + (-1)^{(k+1)(l+1)} v \wedge (u \cdot m). \square \end{aligned}$$

We will show the relation

$$u \cdot (v \cdot m) - (-1)^{kl} v \cdot (u \cdot m) = [u, v] \cdot m$$

by induction on l .

First case : $l = -1$

In this case v is a function on X which will be denoted f . We proceed by induction over k . The cases $k = -1$ or $k = 0$ are obvious so that we assume $k \geq 1$. We set $u = \xi_1 \wedge \dots \wedge \xi_{k+1}$ and $u' = \xi_2 \wedge \dots \wedge \xi_{k+1}$.

Using the two previous lemmas and the induction hypothesis, we get the following sequence of equalities :

$$\begin{aligned} & u \cdot (f \cdot m) - (-1)^k f \cdot (u \cdot m) \\ &= -(-1)^k f \cdot (u \cdot m) \\ &= -(-1)^k f \cdot (\xi_1 \wedge (u' \cdot m) + (-1)^k u' \otimes \xi_1 \cdot m) \\ &= -(-1)^k [f, \xi_1] (u' \cdot m) + (-1)^k \xi_1 \wedge (f \cdot (u' \cdot m)) - [f, u'] \otimes \xi_1 \cdot m \\ &= -(-1)^k [f, \xi_1] (u' \cdot m) + (-1)^k \xi_1 \wedge ([f, u'] \cdot m) - [f, u'] \otimes \xi_1 \cdot m \end{aligned}$$

On the hand,

$$[f, u] = [f, \xi_1] u' - \xi_1 \wedge [f, u'].$$

hence,

$$[f, u] \cdot m = [f, \xi_1] u' \cdot m - \xi_1 \wedge ([f, u'] \cdot m) - (-1)^{k+1} [f, u'] \otimes \xi_1 \cdot m.$$

The cas $l = -1$ follows.

Second case : $l=0$

In this case v is an element of E which will be denoted η . We proceed by induction over k . The cases $k = -1$ or $k = 0$ are obvious so that we assume $k \geq 1$. We set $u = \xi_1 \wedge \dots \wedge \xi_{k+1}$ and $u' = \xi_2 \wedge \dots \wedge \xi_{k+1}$.

Using the two previous lemmas, we get the following sequence of equalities :

$$\begin{aligned} & u \cdot (\eta \cdot m) - \eta \cdot (u \cdot m) \\ &= \xi_1 \wedge (u' \cdot (\eta \cdot m)) + (-1)^k u' \otimes \xi_1 \cdot (\eta \cdot m) - \eta \cdot (\xi_1 \wedge (u' \cdot m) + (-1)^k u' \otimes \xi_1 \cdot m) \\ &= \xi_1 \wedge ([u', \eta] \cdot m) + (-1)^k u' \otimes [\xi_1, \eta] \cdot m - [\eta, \xi_1] \wedge (u' \cdot m) - (-1)^k [\eta, u'] \otimes \xi_1 \cdot m. \end{aligned}$$

On the other hand,

$$[u, \eta] = -[\eta, \xi_1] \wedge u' - \xi_1 \wedge [\eta, u'].$$

hence

$$[u, \eta] \cdot m = -[\eta, \xi_1] \wedge (u' \cdot m) - (-1)^k u' \otimes [\eta, \xi_1] \cdot m - (-1)^k [\eta, u'] \otimes \xi_1 \cdot m - \xi_1 \wedge [\eta, u'] \cdot m.$$

Third case : $l \geq 1$

We proceed by induction. We set $v = \eta_1 \wedge \dots \wedge \eta_{k+1}$ and $u' = \eta_2 \wedge \dots \wedge \eta_{k+1}$. Using the previous lemmas and the induction hypothesis, we get the following sequences of equalities :

$$\begin{aligned} & u \cdot (v \cdot m) - (-1)^{kl} v \cdot (u \cdot m) \\ &= u \cdot (\eta_1 \wedge (v' \cdot m) + (-1)^l v' \otimes \eta_1 \cdot m) - (-1)^{kl} \eta_1 \wedge (v' \cdot (u \cdot m)) \\ &\quad - (-1)^{l(k+l)} v' \wedge (\eta_1 \cdot (u \cdot m)) \\ &= (-1)^{k(l+1)} v' \wedge ([u, \eta_1] \cdot m) + (-1)^k \eta_1 \wedge [u, v'] \cdot m + [u, \eta_1] \wedge (v' \cdot m) \\ &\quad + (-1)^l [u, v'] \otimes \eta_1 \cdot m. \end{aligned}$$

On the other hand,

$$[u, v] = [u, \eta_1] \wedge v' + (-1)^k \eta_1 \wedge [u, v']$$

hence

$$\begin{aligned} [u, v] \cdot m &= [u, \eta_1] \wedge (v' \cdot m) + (-1)^{l(k+1)} v' \wedge ([u, \eta_1] \cdot m) + (-1)^k \eta_1 \wedge ([u, v'] \cdot m) \\ &\quad + (-1)^l [u, v'] \otimes \eta_1 \cdot m. \end{aligned}$$

The case $l \geq 1$ follows. \square

3.2 The ${}^E D_{poly}^*$ -module ${}^E D_{poly}^*(M)$

Let M be a $D(E)$ -module. Denote by π the map from $D(E)$ to $End(M)$ determined by the left $D(E)$ -module structure on M . We will use the same notation as in section 2.2. We will also use the map τ_i from $\left(\bigotimes_{\mathcal{O}_X}^{i-1} D(E)\right) \otimes_{\mathcal{O}_X} \left(D(E) \otimes_{\mathcal{O}_X} M\right) \otimes_{\mathcal{O}_X} \left(\bigotimes_{\mathcal{O}_X}^{q+1-i} D(E)\right)$ to $D(E) \otimes_{\mathcal{O}_X}^{q+1} M$ defined by

$$\tau_i(Q_0 \otimes \dots \otimes Q_{i-1} \otimes (Q_i \otimes m) \otimes Q_{i+1} \otimes \dots \otimes Q_q) = Q_0 \otimes \dots \otimes Q_q \otimes m.$$

Let us introduce the complex ${}^E D_{poly}(M)$ of E -polydifferential operators with values in M as follows :

$${}^E D_{poly}(M) = \bigoplus_{k \geq -1} {}^E D_{poly}^k(M)$$

where

$$\begin{aligned} {}^E D_{poly}^{-1}(M) &= M \\ {}^E D_{poly}^k(M) &= D(E) \otimes_{\mathcal{O}_X}^{k+1} M \text{ if } k \geq 0. \end{aligned}$$

Let us define two maps denoted in the same way

$$\begin{aligned} \bullet : {}^E D_{poly}^* \otimes {}^E D_{poly}^*(M) &\rightarrow {}^E D_{poly}^*(M) \\ \bullet : {}^E D_{poly}^*(M) \otimes {}^E D_{poly}^* &\rightarrow {}^E D_{poly}^*(M). \end{aligned}$$

If P and Q are homogeneous elements of ${}^E D_{poly}^*$ of non negative degree respectively $|P|$ and $|Q|$, f is an element of ${}^E D_{poly}^{-1}$ and m is in M , then

$$\begin{aligned} P \bullet (Q \otimes m) &= \sum_{i=0}^{|P|} (-1)^i |Q| \tau_i \left[\left(id^{\otimes i} \otimes \Delta^{(|Q|+1)} \otimes id^{\otimes |P|-i} \right) (P) \cdot (1^{\otimes i} \otimes_{\mathbb{R}} (Q \otimes m) \otimes_{\mathbb{R}} 1^{\otimes |P|-i}) \right] \\ P \bullet m &= \tau_i \left[\sum_{i=0}^{|P|} (-1)^i \left(id^{\otimes i} \otimes \pi \otimes id^{\otimes |P|-i} \right) (P) \cdot (1^{\otimes i} \otimes_{\mathbb{R}} m \otimes_{\mathbb{R}} 1^{\otimes |P|-i}) \right] \\ f \bullet m &= 0 \\ f \bullet (Q \otimes m) &= 0 \\ (Q \otimes m) \bullet P &= Q \bullet P \otimes m \\ m \bullet P &= 0 \\ m \bullet f &= 0. \end{aligned}$$

Note that the second, the third and the fourth equations could be recovered from the first one. The differential, ∂_M , on ${}^E D_{poly}^*(M)$ is given by : for all

$Q \otimes m$ in ${}^E D_{poly}^*(M)$,

$$\begin{aligned}\partial_M(Q \otimes m) &= (1 \otimes 1) \bullet (Q \otimes m) - (-1)^{|Q|} (Q \otimes m) \bullet (1 \otimes 1) \\ &= \partial(Q) \otimes m\end{aligned}$$

where $1 \otimes 1 \in {}^E D_{poly}^1$.

Theorem 3.2.1 ${}^E D_{poly}^*(M)$ is endowed with a ${}^E D_{poly}^*$ -module structure as follows.

$$\begin{aligned}\forall P \in {}^E D_{poly}^p, \forall (Q \otimes m) \in {}^E D_{poly}^q(M) \\ P \cdot_G (Q \otimes m) = P \bullet (Q \otimes m) - (-1)^{pq} (Q \otimes m) \bullet P\end{aligned}$$

Proof of the theorem :

Let $P \in {}^E D_{poly}^p$, $Q \in {}^E D_{poly}^q$, $\lambda \in {}^E D_{poly}^r(M)$. Introduce the following quantity

$$A(P, Q, \lambda) = (P \bullet Q) \bullet \lambda - P \bullet (Q \bullet \lambda).$$

The theorem follows from the lemma below.

Lemma 3.2.2 *The following equality holds :*

$$A(P, Q, \lambda) = (-1)^{qr} A(P, \lambda, Q)$$

This lemma is well known in the case where $E = TX$ and $M = \mathcal{O}_X$ (see for example the article of Keller in [BCKT]).

In the general case, it follows from a straightforward but tedious computation. \square

3.3 The Hochschild-Kostant-Rosenberg theorem

Theorem 3.3.1 *The map U_{HKR}^M from $({}^E T_{poly}^*(M), 0)$ to $({}^E D_{poly}^*(M), \partial_M)$ defined*

by : for all v_1, \dots, v_n in E and all m in M ,

$$U_{HKR}^M(v_0 \wedge \dots \wedge v_n \otimes m) = \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} \epsilon(\sigma) v_{\sigma(0)} \otimes \dots \otimes v_{\sigma(n)} \otimes m$$

$$U_{HKR}^M(m) = m$$

is a quasi-isomorphism.

The first one to have proved such a statement in the affine case (for $E = TX$ and $M = \mathcal{O}_X$) seems to be J. Vey ([V]). A proof for the tangent bundle of any manifold (and $M = \mathcal{O}_X$) can be found in [Ko]. This theorem is proved in [C1] for any Lie algebroid and $M = \mathcal{O}_X$.

Proof of the theorem :

This theorem will be a consequence of the proof of theorem 3.4.1 and of the following well known result.

Lemma 3.3.2 *T be a finite dimensional \mathbb{R} -vector space. Consider the complex $\wedge^* T = \bigoplus_{p \in \mathbb{N}} \wedge^p T$ with zero differential and the complex $\bigoplus_{p \in \mathbb{N}} \left(\bigotimes^p S(E) \right)$ with the differential*

$$\partial = id^{\otimes p} \otimes 1 + (-1)^{p-1} 1 \otimes id^{\otimes p} + (-1)^{p-1} \sum_{i=0}^n (-1)^i id^{\otimes i} \otimes \Delta \otimes id^{\otimes n-i}.$$

The \mathbb{R} -linear map Θ from $\wedge^ T$ to $\bigoplus_{p \in \mathbb{N}} \bigotimes^p S(T)$ defined by : for all v_1, \dots, v_p in T ,*

$$\begin{aligned} \Theta(v_0 \wedge \dots \wedge v_p) &= \frac{1}{(p+1)!} \sum_{\sigma \in S_{p+1}} \epsilon(\sigma) v_{\sigma(0)} \otimes \dots \otimes v_{\sigma(p)} \\ \Theta(1) &= 1 \end{aligned}$$

is a quasi-isomorphism.

3.4 Main statement

We have seen that $\Gamma \left({}^E D_{poly}^*(M) \right)$ is a module over the DGLA $\Gamma \left({}^E D_{poly}^* \right)$. As we know ([C1]) that there is a L_∞ -morphism from $\Gamma \left({}^E T_{poly}^* \right)$ to $\Gamma \left({}^E D_{poly}^* \right)$, we deduce that $\Gamma \left({}^E D_{poly}^*(M) \right)$ is naturally endowed with the structure of an L_∞ -module over the DGLA $\Gamma \left({}^E T_{poly}^* \right)$. We can now state the main result of this paper.

Theorem 3.4.1 *There is a quasi-isomorphism of L_∞ -modules over $\Gamma \left({}^E T_{poly}^* \right)$ from $\Gamma \left({}^E T_{poly}^*(M) \right)$ to $\Gamma \left({}^E D_{poly}^*(M) \right)$ that induces U_{HKR}^M in cohomology.*

Our result extends Calaque's result ([C1], take $M = \mathcal{O}_X$) and Kontsevitch's result ([Ko], take $M = \mathcal{O}_X$ and $E = TX$).

4 Proof

The proof is analogous to that of [D1], [C1], [D2], [CDH].

4.1 Fedosov Resolutions

As before, E will denote a Lie algebroid and M will be a $D(E)$ -module.

Following Fedosov and Dolgushev ([Fe], [D1]), Calaque introduced ([C1], see also [CDH]), the locally free \mathcal{O}_X -modules $\mathcal{W} = \widehat{S}(E^*)$, \mathcal{T}^* and \mathcal{D}^* . Let us recall their definition.

- $\mathcal{W} = \widehat{S}(E^*)$ is the locally free \mathcal{O}_X -module whose sections are functions that are formal in the fiber. An element s of $\Gamma(U, \mathcal{W})$ can be locally written

$$s = \sum_{l=0}^{\infty} s_{i_1, \dots, i_l} y^{i_1} \dots y^{i_l}$$

where y^1, \dots, y^d are coordinates in the fiber of E and s_{i_1, \dots, i_l} are coefficients of a symmetric covariant E -tensor.

- $\mathcal{T}^* = \mathcal{W} \otimes_{\mathcal{O}_X} \wedge^{*+1} E$ is the graded locally free \mathcal{O}_X -module of formal fiberwise polyvector fields on E with shifted degree. A homogeneous section of degree k of \mathcal{T}^* can be locally written

$$\sum_{l=0}^{\infty} v_{i_1, \dots, i_l}^{j_0, \dots, j_k} y^{i_1} \dots y^{i_l} \frac{\partial}{\partial y^{j_0}} \wedge \dots \wedge \frac{\partial}{\partial y^{j_k}}$$

where $v_{i_1, \dots, i_l}^{j_0, \dots, j_k}$ are components of an E -tensor symmetric covariant in the indices i_1, \dots, i_l , contravariant antisymmetric in the indices j_0, \dots, j_k

- $\mathcal{D}^* = \widehat{S}(E^*) \otimes_{\mathcal{O}_X} T^{*+1}(S(E))$ is the graded locally free \mathcal{O}_X -module of formal fiberwise E -polydifferential operators with shifted degree. A homogeneous section of degree k of \mathcal{D}^* can be locally written

$$\sum_{l=0}^{\infty} P_{i_1, \dots, i_l}^{\alpha_0, \dots, \alpha_k}(x) y^{i_1} \dots y^{i_l} \frac{\partial^{|\alpha_0|}}{\partial y^{\alpha_0}} \otimes \dots \otimes \frac{\partial^{|\alpha_k|}}{\partial y^{\alpha_k}}$$

where the α_i 's are multi-indices, the $P_{i_1, \dots, i_l}^{\alpha_0, \dots, \alpha_k}(x)$ are components of an E -tensor with obvious symmetry.

We will need to introduce the \mathcal{O}_X -modules $\mathcal{D}^*(M)$ and $\mathcal{T}^*(M)$.

• $\mathcal{T}^*(M)$ is the graded \mathcal{O}_X -module of formal fiberwise polyvector fields on E with values in M with shifted degree. A homogeneous section of degree k of $\mathcal{T}^*(M)$ can be locally written

$$\sum_{l=0}^{\infty} m_{i_1, \dots, i_l}^{j_0, \dots, j_k} y^{i_1} \dots y^{i_l} \frac{\partial}{\partial y^{j_0}} \wedge \dots \wedge \frac{\partial}{\partial y^{j_k}}$$

where $m_{i_1, \dots, i_l}^{j_0, \dots, j_k}$ are components of an E -tensor with values in M symmetric covariant in the indices i_1, \dots, i_l , contravariant antisymmetric in the indices j_0, \dots, j_k .

• $\mathcal{D}^*(M)$ is the graded \mathcal{O}_X -modules of formal fiberwise E -polydifferential operators with values in M (with shifted degree). A homogeneous section of degree k of $\mathcal{D}^*(M)$ can be locally written

$$\sum_{l=0}^{\infty} \mu_{i_1, \dots, i_l}^{\alpha_0, \dots, \alpha_k}(x) y^{i_1} \dots y^{i_l} \frac{\partial^{|\alpha_0|}}{\partial y^{\alpha_0}} \otimes \dots \otimes \frac{\partial^{|\alpha_k|}}{\partial y^{\alpha_k}}$$

where the α_i 's are multi-indices, the $\mu_{i_1, \dots, i_l}^{\alpha_0, \dots, \alpha_k}(x)$ are coefficients of an E -tensor with values in M with obvious symmetry.

Remark :

One has the obvious equality $\mathcal{T}^*(\mathcal{O}_X) = \mathcal{T}^*$ and $\mathcal{D}^*(\mathcal{O}_X) = \mathcal{D}^*$.

Notation :

Let \mathbb{R}_{formal}^d be the formal completion of \mathbb{R}^d at the origin. The ring of functions on \mathbb{R}_{formal}^d is $\mathbb{R}[[y^1, \dots, y^d]]$ and the Lie-Rinehart algebra of vector fields is $Der(\mathbb{R}[[y^1, \dots, y^d]])$. Denote by $T_{poly}^*(\mathbb{R}_{formal}^d)$ and $D_{poly}^*(\mathbb{R}_{formal}^d)$ the DGLAs of polyvector fields and polydifferential operators on \mathbb{R}_{formal}^d respectively. If $t_1 \in D_{poly}^{k_1-1}(\mathbb{R}_{formal}^d)$ and $t_2 \in D_{poly}^{k_2-1}(\mathbb{R}_{formal}^d)$, one defines their cup-product $t_1 \sqcup t_2 \in D_{poly}^{k_1+k_2-1}(\mathbb{R}_{formal}^d)$ by :

$$\begin{aligned} \forall a_1, \dots, a_{k_1+k_2} &\in \mathbb{R}[[y^1, \dots, y^d]], \\ (t_1 \sqcup t_2)(a_1, \dots, a_{k_1+k_2}) &= t_1(a_1, \dots, a_{k_1}) t_2(a_{k_1+1}, \dots, a_{k_1+k_2}) \end{aligned}$$

The cup-product endows $D_{poly}^*(\mathbb{R}_{formal}^d)$ with the structure of a DGAA.

Remark:

Fiberwise product endows \mathcal{W} with the structure of bundle of commutative algebra. \mathcal{T}^* is a differential Lie algebra with zero differential and Lie bracket induced by fiberwise Schouten bracket on $T_{poly}^*(\mathbb{R}_{formal}^d)$. Similarly, fiberwise Schouten bracket allows to endow $\mathcal{T}^*(M)$ with a \mathcal{T}^* -module structure. We can make the same type of remark for \mathcal{D} , $\mathcal{D}(M)$ and the Gerstenhaber bracket.

Let \mathcal{B} be any of the \mathcal{O}_X -modules introduced above. We will need to tensor \mathcal{B} by $\wedge^*(E^*)$. We set ${}^E\Omega(\mathcal{B}) = \wedge^*(E^*) \otimes \mathcal{B}$.

Structures on ${}^E\Omega(\mathcal{B})$

- ${}^E\Omega(\mathcal{W})$ is a bundle of graded commutative algebras with grading given by exterior degree of E -forms.

- Schouten bracket on $T_{poly}^*(\mathbb{R}_{formal}^d)$ induces a structure of sheaf of graded Lie algebras over ${}^E\Omega^*(\mathcal{T})$. The grading is the sum of the exterior degree and the degree of an E -polyvector. Fiberwise Schouten bracket also endows ${}^E\Omega^*(\mathcal{T}(M))$ with structure of module over the graded Lie algebra ${}^E\Omega^*(\mathcal{T})$. These structures will be respectively denoted by $[\cdot, \cdot]_S$ and \cdot_S . By fiberwise exterior product on $T_{poly}^*(\mathbb{R}_{formal}^d)$, ${}^E\Omega^*(\mathcal{T})$ also carries a structure of sheaf of graded commutative algebras and ${}^E\Omega^*(\mathcal{T}(M))$ becomes a module over the sheaf of graded commutative algebras ${}^E\Omega^*(\mathcal{T})$. These structures will both be denoted by a \wedge . Thus ${}^E\Omega^*(\mathcal{T}(M))$ is a module over the sheaf of Gerstenhaber algebras ${}^E\Omega(\mathcal{T})$.

- Using fiberwise Gerstenhaber bracket, we see that ${}^E\Omega^*(\mathcal{D})$ is a sheaf of differential graded Lie algebras and ${}^E\Omega(\mathcal{D}(M))$ is a module over the sheaf of DGLAs ${}^E\Omega(\mathcal{D})$. These two structures will be denoted $[\cdot, \cdot]_G$ and \cdot_G . The grading is the sum of the exterior degree and the degree of the E -polydifferential operator. Cuproduct in the space $D_{poly}^*(\mathbb{R}_{formal}^d)$ endows ${}^E\Omega(\mathcal{D})$ with the structure of a sheaf of DGAAAs and ${}^E\Omega(\mathcal{D}(M))$ with the structure of a module over the sheaf of DGAAAs ${}^E\Omega(\mathcal{D})$.

${}^E\Omega(\mathcal{W})$, ${}^E\Omega(\mathcal{T}(M))$ and ${}^E\Omega(\mathcal{D}(M))$ are equipped with a decreasing filtration given by the order of the monomials in the fiber coordinates y^i .

In the sequel, we will denote by ξ^i the variable y^i considered as an element of $\wedge^1(E^*)$. Introduce the 2-nilpotent derivation $\delta : {}^E\Omega^*(\mathcal{W}) \rightarrow {}^E\Omega^{*+1}(\mathcal{W})$ of the sheaf of super algebras ${}^E\Omega^*(\mathcal{W})$ defined by $\delta = \xi^i \frac{\partial}{\partial y^i}$. Using \cdot_S and

\cdot_G, δ extends to a 2-nilpotent differential of $\mathcal{T}(M)$ and $\mathcal{D}(M)$.

Proposition 4.1.1 *Let \mathcal{B} be any of the sheaves \mathcal{W} , $\mathcal{T}(M)$ or $\mathcal{D}(M)$.*

$$H^{\geq 1} \left({}^E\Omega(\mathcal{B}), \delta \right) = 0.$$

Furthermore, we have the following isomorphisms of sheaves of graded \mathcal{O}_X -modules :

$$\begin{aligned} H^0 \left({}^E\Omega(\mathcal{W}), \delta \right) &= \mathcal{O}_X \\ H^0 \left({}^E\Omega(\mathcal{T}(M)), \delta \right) &= {}^E T_{poly}(M) \\ H^0 \left({}^E\Omega(\mathcal{D}^*(M)), \delta \right) &= \otimes^{*+1}_{\mathcal{O}_X} S(E) \otimes M. \end{aligned}$$

This proposition is known for \mathcal{W} and $M = \mathcal{O}_X$. It is due to Dolgushev ([D1]) for $E = TX$ and to Calaque ([C1]) for any Lie algebroid. Our proof is totally analogous to that of Dolgushev.

Proof of the proposition :

Let us consider the operator $\kappa : {}^E\Omega^*(\mathcal{B}) \rightarrow {}^E\Omega^{*-1}(\mathcal{B})$ defined by

$$\begin{aligned} \forall \sigma \in \Omega^{>0}(\mathcal{T}(M)), \quad \kappa(\sigma) &= y^m \frac{\partial}{\partial \xi^m} \int_0^1 \sigma(x, ty, t\xi) \frac{dt}{t} \\ \kappa|_{\mathcal{T}(M)} &= 0. \end{aligned}$$

It satisfies the relation

$$\delta\kappa + \kappa\delta + \mathcal{H} = id$$

where

$$\forall u \in {}^E\Omega^*(\mathcal{B}), \quad \mathcal{H}(u) = u|_{y^i=\xi^i=0}.$$

The proposition follows. \square

Remark :

We will keep using the operator κ in our proofs. Note that κ has the two following properties :

- $\kappa^2 = 0$.
- κ increases the filtration in the variables y^i 's by one.

Let ∇ be a torsion free connection on E . Let (e_1, \dots, e_n) be a local basis of E . Denote by $\Gamma_{i,j}^k$ the Christoffel symbol of ∇ with respect to this basis. As it is explained in previous works ([D1], [C1], [D2], [CDH]) such a connection allows to define a connection on \mathcal{W} (still denoted ∇) as follows :

$$\nabla = {}^E d + \Gamma \cdot \quad \text{with} \quad \Gamma = -\xi^i \Gamma_{i,j}^k y^j \frac{\partial}{\partial y^k}.$$

It also allows to define a connection on $\mathcal{T}(M)$ and $\mathcal{D}(M)$ given by

$$\nabla_M = {}^E d_M + \Gamma \cdot.$$

For example, if $\sigma = \sum_{l=0}^{\infty} m_{i_1, \dots, i_l}^{j_0, \dots, j_k} y^{i_1} \dots y^{i_l} \frac{\partial}{\partial y^{j_0}} \wedge \dots \wedge \frac{\partial}{\partial y^{j_k}}$ is a local section of $\mathcal{T}(M)$, one has

$$\begin{aligned} \nabla_M(\sigma) &= \sum_{l=0}^{\infty} {}^E d_M(m_{i_1, \dots, i_l}^{j_0, \dots, j_k}) y^{i_1} \dots y^{i_l} \frac{\partial}{\partial y^{j_0}} \wedge \dots \wedge \frac{\partial}{\partial y^{j_k}} \\ &+ \sum_{l=0}^{\infty} m_{i_1, \dots, i_l}^{j_0, \dots, j_k} \Gamma \cdot_S \left(y^{i_1} \dots y^{i_l} \frac{\partial}{\partial y^{j_0}} \wedge \dots \wedge \frac{\partial}{\partial y^{j_k}} \right). \end{aligned}$$

Since ∇ is torsion free, one has $\nabla_M \delta + \delta \nabla_M = 0$. The curvature tensor allows to define the following element of ${}^E \Omega^2(\mathcal{T}^0)$

$$R = -\frac{1}{2} \xi^i \xi^j (R_{ij})_k^l(x) y^k \frac{\partial}{\partial y^l}.$$

A computation shows $\nabla_M^2 = R \cdot : {}^E \Omega^*(\mathcal{B}) \rightarrow {}^E \Omega^{*+2}(\mathcal{B})$.

Theorem 4.1.2 *Let \mathcal{B} be any of the sheaves $\mathcal{T}(M)$ and $\mathcal{D}(M)$. There exists a section*

$$A = \sum_{s=2}^{\infty} \xi^k A_{k, i_1, \dots, i_s}^j(x) y^{i_1} \dots y^{i_s} \frac{\partial}{\partial y^j}$$

of the sheaf ${}^E \Omega^1(\mathcal{T}^0)$ such that the operator $D_M : {}^E \Omega^(\mathcal{B}) \rightarrow {}^E \Omega^{*+1}(\mathcal{B})$*

$$D_M = \nabla_M - \delta + A \cdot$$

is 2-nilpotent and is compatible with the DG-algebraic structures on ${}^E \Omega^(\mathcal{B})$.*

The theorem was proved for $\mathcal{B} = \mathcal{W}, \mathcal{T}$ and \mathcal{D} in [D1] for $E = TX$ and in [C1] for any algebroid. Our proof is inspired by that of [D1] (see also [C1]).

Proof of the theorem

A computation shows that D_M is two-nilpotent if the following condition holds :

$$R + \nabla A - \delta A + \frac{1}{2}[A, A]_S = 0. \quad (1)$$

The following equation

$$A = \kappa R + \kappa \left(\nabla(A) + \frac{1}{2}[A, A]_S \right) \quad (2)$$

has a unique solution (computed by induction on the order in the fiber coordinates y^i 's). It is shown in [D1] that the solution of the equation (2) satisfies (1). We won't reproduce the proof here.

If α is in ${}^E\Omega(\mathcal{T})$ and μ is in ${}^E\Omega(\mathcal{T}(M))$, we have the relations

$$\begin{aligned} D(\alpha \wedge \mu) &= D(\alpha) \wedge \mu + (-1)^{|\alpha|+1} \alpha \wedge D(\mu) \\ D(\alpha \cdot_S \mu) &= D(\alpha) \cdot_S \mu + (-1)^{|\alpha|} \alpha \cdot_S D(\mu) \end{aligned}$$

where $|\alpha|$ denotes the degree of α in the graded Lie algebra ${}^E\Omega(\mathcal{T})$. Similarly, if α is in ${}^E\Omega(\mathcal{D})$ and μ is in ${}^E\Omega(\mathcal{D}(M))$, we have the relations

$$\begin{aligned} D(\alpha \sqcup \mu) &= D(\alpha) \sqcup \mu + (-1)^{|\alpha|+1} \alpha \sqcup D(\mu) \\ D(\alpha \cdot_G \mu) &= D(\alpha) \cdot_G \mu + (-1)^{|\alpha|} \alpha \cdot_G D(\mu) \end{aligned}$$

where $|\alpha|$ denotes the degree of α in the graded Lie algebra ${}^E\Omega(\mathcal{D})$. \square

One can compute the cohomology of the Fedosov differential D .

Theorem 4.1.3 *Let \mathcal{B} be any of the sheaves ${}^E\Omega(\mathcal{W})$, ${}^E\Omega(\mathcal{T}(M))$ or ${}^E\Omega(\mathcal{D}(M))$,*

$$H^{\geq 1}(\mathcal{B}, D) = 0.$$

Furthermore, we have the following isomorphisms of sheaves of graded commutative algebras

$$\begin{aligned} H^0({}^E\Omega(\mathcal{W}), D) &\simeq \mathcal{O}_X \\ H^0({}^E\Omega(\mathcal{T}), D) &\simeq \wedge^{*+1} E \end{aligned}$$

and the following isomorphism of sheaves of DGAA's (over \mathbb{R})

$$H^0({}^E\Omega(\mathcal{D}), D) \simeq \otimes^{*+1} S(E).$$

Using the identification above, $H^0\left({}^E\Omega(\mathcal{T}(M)), D\right)$ and $\bigwedge^{*+1} E \otimes_{\mathcal{O}_X} M$ are isomorphic as $H^0\left({}^E\Omega(\mathcal{T}), D\right) \simeq \bigwedge^{*+1} E$ -modules. Furthermore $H^0\left({}^E\Omega(\mathcal{D}(M)), D\right)$ and $\otimes^{*+1} S(E) \otimes_{\mathcal{O}_X} M$ are isomorphic as $H^0\left({}^E\Omega(\mathcal{D}), D\right) \simeq \otimes^{*+1} S(E)$ -modules.

This theorem is already known for $M = \mathcal{O}_X$: see [D1] for the case where $E = TX$ and [C1], [C2] for any Lie algebroid. The proof of the theorem is very similar to the proof in the case where $M = \mathcal{O}_X$. That is why, we give only a sketch of it and refer to [CDH] and [C2] for details.

Proof of the theorem :

The first assertion of the theorem follows from a spectral sequence argument using the filtration on \mathcal{B} given by the order on the y^i 's (see [CDH], theorem 2.4 for details).

Let $u \in \mathcal{B} \cap \text{Ker} \delta$. One can show (solving the equation by induction on the order in the fiber coordinates y^i 's) that there exists a unique $\lambda(u) \in \mathcal{B} \cap \text{Ker} D$ such that

$$\lambda(u) = u + \kappa(\nabla \lambda(u) + A \cdot \lambda(u)).$$

Thus, we have defined a map $\lambda : \text{Ker} \delta \cap \mathcal{B} \rightarrow \text{Ker} D \cap \mathcal{B}$. One can show that λ is bijective and that $\lambda^{-1} = \mathcal{H}$. The following relations (easy to establish) allows to finish the proof of the theorem :

- If $\alpha, \beta \in {}^E\Omega(\mathcal{W})$, then $\mathcal{H}(\alpha\beta) = \mathcal{H}(\alpha)\mathcal{H}(\beta)$
- If $\alpha \in {}^E\Omega(\mathcal{T})$ and $\mu \in {}^E\Omega(\mathcal{T}(M))$, then $\mathcal{H}(\alpha \wedge \mu) = \mathcal{H}(\alpha) \wedge \mathcal{H}(\mu)$.
- If $\alpha \in {}^E\Omega(\mathcal{D})$ and $\mu \in {}^E\Omega(\mathcal{D}(M))$, then $\mathcal{H}(\alpha \sqcup \mu) = \mathcal{H}(\alpha) \sqcup \mathcal{H}(\mu)$. \square

As D is compatible with the action \cdot_S of ${}^E\Omega^*(\mathcal{T})$ over ${}^E\Omega^*(\mathcal{T}(M))$ and hence with the Schouten bracket on ${}^E\Omega^*(\mathcal{T})$, $H^*\left({}^E\Omega(\mathcal{T}), D\right)$ is a graded Lie algebra and $H^*\left({}^E\Omega(\mathcal{T}(M)), D\right)$ is a module over the graded Lie algebra $H^*\left({}^E\Omega^*(\mathcal{T}), D\right)$. So, it is natural to wonder whether the isomorphisms of the previous proposition respect this structure.

Proposition 4.1.4 *The map $\mathcal{H} : \mathcal{T}^* \cap \text{Ker} D \rightarrow \mathcal{T}^* \cap \text{Ker} \delta \simeq {}^E T_{poly}^*$ is an isomorphism of graded Lie algebras.*

The map $\mathcal{H} : \mathcal{T}^*(M) \cap \text{Ker} D \rightarrow {}^E T_{poly}^*(M)$ is an isomorphism of modules over the graded Lie algebras $\mathcal{T}^* \cap \text{Ker} D \simeq {}^E T_{poly}^*$.

Proof of the proposition :

The first assertion of the proposition is proved in [C1], [C2]. Let us now prove the second assertion. Denote by π the map from $D(E)$ to $\text{End}(M)$ defined by the action of $D(E)$ on M .

Let m be an element of M and let $u = \sum_{i=1}^d u_i(x) e_i \in {}^E T_{poly}^0$. Using the definition of λ , one finds easily :

$$\begin{aligned} \lambda(m) &= m + \sum_{i=1}^d y^i \pi(e_i) \cdot m \mod |y| \\ \lambda(u) &= \sum_{i=1}^d u_i \frac{\partial}{\partial y^i} \mod |y| \end{aligned}$$

Hence

$$\lambda(u) \cdot \lambda(m) = \sum_{i=1}^d u_i \pi(e_i) \cdot m \mod |y|.$$

and

$$\mathcal{H}(\lambda(u) \cdot \lambda(m)) = u \cdot m = \mathcal{H}(\lambda(u)) \cdot \mathcal{H}(\lambda(m)).$$

The end of the proof follows from the definition of the action of ${}^E T_{poly}$ on ${}^E T_{poly}^*(M)$ and the previous theorem. \square

The morphism μ'_M

Let us first recall the construction of μ' ([CDH]). \mathcal{T}^0 is the sheaf of Lie algebras over the sheaf of algebras $\mathcal{T}^{-1} = \widehat{S}(E^*)$ and we have $\mathcal{D}^0 = D(\mathcal{T}^0)$. The morphism of Lie algebras $\lambda = \mathcal{H}^{-1} : E \rightarrow \mathcal{T}^0 \cap \text{Ker} D$ induces a morphism of sheaves of algebras $\mu : D(E) \rightarrow \mathcal{D}^0$ that takes values in $\text{Ker} D \cap \mathcal{D}^0$. We will denote by μ' the only morphism of sheaves of DGAAs from ${}^E D_{poly}^*$ to \mathcal{D}^* defined by

$$\mu'|_{E D_{poly}^0} = \mu, \quad \mu'|_{\mathcal{O}_X} = \lambda.$$

Let $\mu'_M : {}^E D_{poly}^*(M) \rightarrow \mathcal{D}^*(M)$ the morphism defined by : $\forall P_0, \dots, P_n \in D(E), \forall m \in M$

$$\begin{aligned} \mu'_M(m) &= \lambda(m) \\ \mu'_M(P_0 \otimes \dots \otimes P_n \otimes m) &= \mu(P_0) \otimes \dots \otimes \mu(P_n) \otimes \lambda(m) \end{aligned}$$

Note that $\mu' = \mu'_{\mathcal{O}_X}$.

Proposition 4.1.5 *a) μ is an isomorphism of sheaves of algebras from $D(E)$ to $\mathcal{D}^0 \cap \text{Ker} D$. It is also a morphism of sheaves of bialgebroids.*

b) μ' is an isomorphism of sheaves of DGLAs from ${}^E D_{poly}^$ to $\mathcal{D}^* \cap \text{Ker} D$. It is also an isomorphism of sheaves of DGAAAs.*

c) $\mu'_M : {}^E D_{poly}^(M) \rightarrow \mathcal{D}^*(M) \cap \text{Ker} D$ is an isomorphism of modules over the sheaf of DGLAs ${}^E D_{poly}^* \simeq \mathcal{D}^* \cap \text{Ker} D$. It is also an isomorphism of modules over the sheaf of DGAAAs ${}^E D_{poly}^* \simeq \mathcal{D}^* \cap \text{Ker} D$.*

Proof of the proposition :

a) and b) are shown in [CDH]. The proof of c) is analogous. Using the definition of μ and μ'_M , one can show easily the following :

$$\forall P \in D(E), \forall m \in M, \mu'_M(P \cdot m) = \mu(P) \cdot \mu'_M(m).$$

As moreover μ is an isomorphism of bialgebroids ([CDH]), μ'_M is a morphism of modules over the sheaf of DGLAs ${}^E D_{poly}^* \simeq \mathcal{D}^* \cap \text{Ker} D$. μ'_M is clearly a morphism of modules over the sheaf of DGAAAs ${}^E D_{poly}^* \simeq \mathcal{D}^* \cap \text{Ker} D$. The fact that μ'_M is an isomorphism follows from a) and theorem 4.1.3. \square

4.2 Kontsevitch's result

Recall that \mathbb{R}_{formal}^d is the formal completion of \mathbb{R}^d at the origin. The ring of functions on \mathbb{R}_{formal}^d is $\mathbb{R}[[y^1, \dots, y^d]]$ and the Lie-Rinehart algebra of vector fields is $Der(\mathbb{R}[[y^1, \dots, y^d]])$. Denote by $T_{poly}^*(\mathbb{R}_{formal}^d)$ and $D_{poly}^*(\mathbb{R}_{formal}^d)$ the DGLAs of polyvector fields and polydifferential operators on \mathbb{R}_{formal}^d respectively.

Theorem 4.2.1 *There exists a quasi-isomorphism of L_∞ algebras U from $T_{poly}^*(\mathbb{R}_{formal}^d)$ to $D_{poly}^*(\mathbb{R}_{formal}^d)$ such that*

- (1) *The first structure map $U^{[1]}$ is the quasi-isomorphism U_{HKR} .*
- (2) *U is $GL_d(\mathbb{R})$ -equivariant.*
- (3) *If $n > 1$ then, for any vector fields $v_1, \dots, v_n \in T_{poly}^0(\mathbb{R}_{formal}^d)$*

$$U^{[n]}(v_1, \dots, v_n) = 0$$

- (4) *If $n > 1$ then for any vector field v linear in the coordinates y^i and polyvector fields $\chi_2, \dots, \chi_n \in T^*(\mathbb{R}_{formal}^d)$*

$$U^{[n]}(v, \chi_2, \dots, \chi_n) = 0.$$

Moreover, Kontsevitch gives an explicit expression for $U^{[n]}$ ([Ko], see also [AMM] or [BCKT] for a detailed exposition) which involves admissible graphs.

Definition 4.2.2 *Let n and m be two integers. An admissible graph Γ of type (n, m) is a labeled oriented graph satisfying the following properties. Let V_Γ be the set of vertices of Γ and E_Γ be the set of edges of Γ .*

- 1) $V_\Gamma = \{1, \dots, n\} \sqcup \{\bar{1}, \dots, \bar{m}\}$. Elements of $\{1, \dots, n\}$ are called first type vertices and element of $\{\bar{1}, \dots, \bar{m}\}$ second type vertices.
- 2) Every edge of Γ starts from a first type vertex.
- 3) There is no loop.
- 4) Two edges can't have the same source and the same target.

We will write $G_{n,m}$ for the set of admissible graphs with n first type vertices and m second type vertices. Let Γ be an element of $G_{n,m}$. We will denote by E_Γ the set of its edges. If γ is in E_Γ , then $s(\gamma)$ will be its source and $t(\gamma)$ its target. Let us introduce the following notation : if k is a vertex of first type

$$(k, *) = \{\gamma \in E_\Gamma \mid s(\gamma) = k\} = \{e_k^1, \dots, e_k^{s_k}\}$$

Similarly, the subset $(*, k)$ of E_Γ is defined for any vertex of Γ .

Let $\alpha_1, \dots, \alpha_n$ be n polyvector fields such that for any $j \in [1, n]$, α_j is a s_j polyvector fields. We will associate to such $\alpha_1, \dots, \alpha_n$ an m polydifferential operator $B_\Gamma(\alpha_1, \dots, \alpha_n)$. Write

$$\alpha_j = \sum_{i_1, \dots, i_{s_j}} \alpha^{i_1, \dots, i_{s_j}} \partial_{i_1} \wedge \dots \wedge \partial_{i_{s_j}} \text{ with } \partial_k = \frac{\partial}{\partial y^k}.$$

If $I : E_\Gamma \rightarrow \{1, \dots, d\}$ is a map from E_Γ to $\{1, \dots, d\}$, we set

$$D_{I(x)} = \prod_{e \in (*, x)} \partial_{I(e)} \\ \alpha_k^I = \alpha_k^{I(e_k^1), \dots, I(e_k^{s_k})}.$$

$B_\Gamma(\alpha_1 \otimes \dots \otimes \alpha_n)$ is the m -differential operator defined by : for any functions f_1, \dots, f_m ,

$$B_\Gamma(\alpha_1 \otimes \dots \otimes \alpha_n)(f_1, \dots, f_m) = \sum_{I: E_\Gamma \rightarrow \{1, \dots, d\}} \prod_{k=1}^{k=n} D_{I(k)} \alpha_k^I \prod_{l=1}^{l=m} D_{I(\bar{l})} f_l$$

If $\alpha_1, \dots, \alpha_n$ are any graded elements of T_{poly} , one has

$$U^{[n]}(\alpha_1, \dots, \alpha_n) = \sum_{\Gamma \in G_{n,m}} W_{\Gamma} B_{\Gamma}(\alpha_1 \otimes \dots \otimes \alpha_n)$$

where the sum is taken over the graph Γ in $G_{n,m}$ such that $B_{\Gamma}(\alpha_1 \otimes \dots \otimes \alpha_n)$ is defined and the relation $m - 2 + 2n = \sum_{i=1}^n s_k$ is satisfied. The coefficient W_{Γ} can be different from zero only if $|E_{\Gamma}| = 2n + m - 2$. Let us now describe it.

Let \mathcal{H} be the Poincaré half plane :

$$\mathcal{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}.$$

Introduce

$$Conf_{n,m} = \{(p_1, \dots, p_n, q_{\bar{1}}, \dots, q_{\bar{m}}) \in \mathcal{H}^n \times \mathbb{R}^m \mid p_i \neq p_j, q_{\bar{i}} \neq q_{\bar{j}}\}.$$

The group $G = \{z \mapsto az + b \mid (a, b) \in \mathbb{R}^{+*} \times \mathbb{R}\}$ acts freely on $Conf_{n,m}$. The quotient $C_{n,m} = Conf_{n,m}/G$ is a manifold of dimension $2n + m - 2$. As $Conf_{n,m}$ is naturally oriented and the action of G preserves this orientation, $C_{n,m}$ inherits a natural orientation. $C_{n,m}$ has several connected components, we will use one of them $C_{n,m}^+$ defined by

$$C_{n,m}^+ = \{(p_1, \dots, p_n, q_{\bar{1}}, \dots, q_{\bar{m}}) \mid q_{\bar{1}} < \dots < q_{\bar{m}}\}.$$

If $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n\} \sqcup \{\bar{1}, \dots, \bar{m}\}$ (with $i \neq j$), one defines a function

$$\begin{aligned} \theta_{i,j} : C_{n,m} &\rightarrow \mathbb{R}/2\pi\mathbb{Z} \\ (z_k)_{k \in [1,n] \sqcup [1,\bar{m}]} &\mapsto \frac{1}{2\pi} \text{Arg} \frac{z_j - z_i}{z_j - \bar{z}_i} . \end{aligned}$$

Let Γ be an element of $G_{n,m}$. We order E_{Γ} with the lexicographic order and define the closed form

$$\omega_{\Gamma} = \bigwedge_{\gamma \in E_{\Gamma}} d\theta_{s(\gamma), t(\gamma)}.$$

One then put

$$W_{\Gamma} = \int_{C_{n,m}^+} \omega_{\Gamma}.$$

This integral is absolutely convergent as the integrand extends to a differential form on a compactification of $C_{n,m}^+$, $\bar{C}_{n,m}^+$, which is a manifold with corners of dimension $2n + m - 2$ ([Ko], see also [AMM] and [BCKT]).

Lemma 4.2.3 *Let n be a non zero integer. For any polyvectorfields $\gamma_1, \dots, \gamma_n$, one has*

$$U^{[n+1]}(\frac{\partial}{\partial y^i}, \gamma_1, \dots, \gamma_n) = 0.$$

Proof of the lemma :

We will prove that for any Γ in $G_{n+1,m}$ having a contribution in $U^{[n+1]}$, one has $W_\Gamma = 0$. For such a Γ , there is no edge going to the vertex 1 and there is exactly one edge starting from the vertex 1 and going to a vertex i_0 which might be of first or of second type. We will denote by Γ' the element of $G_{n,m}$ obtained from Γ by removing the vertex 1 and the edge going from 1 to i_0 .

First case : i_0 is of first type

Using the action of G , we put p_{i_0} in i . If j is in $[1, n+1] - \{i_0\}$, we will write $z_j = a_j + ib_j$ for the affix of p_j and if k is in $[1, m]$, we will write t_k for the coordinate of q_k . One has

$$\omega_\Gamma = \frac{1}{2\pi} dArg \left(\frac{i - z_1}{i - \bar{z}_1} \right) \wedge \omega_{\Gamma'}$$

and $\omega_{\Gamma'}$ is a differential form of degree $2(n+1)+m-3$ in the $2(n-1)+m$ variables $da_2, db_2, \dots, \widehat{da_{i_0}}, \widehat{db_{i_0}}, \dots, da_{n+1}, db_{n+1}, dt_1, \dots, dt_m$. Hence $\omega_{\Gamma'} = 0$ and $\omega_\Gamma = 0$.

Second case : i_0 is of second type

We treat the case where $i_0 \neq \bar{m}$. The case where $i_0 = \bar{m}$ is treated analogously. Using the action of G , we put q_{i_0} in 0 and q_{i_0+1} in 1. One has

$$\omega_\Gamma = \frac{1}{\pi} dArg(z_1) \wedge \omega_{\Gamma'}.$$

$\omega_{\Gamma'}$ is a differential form of degree $2(n+1)+m-3$ in the $2n+m-2$ variables $a_2, b_2, \dots, a_{n+1}, b_{n+1}, q_1, \dots, \widehat{q_{i_0}}, \widehat{q_{i_0+1}}, \dots, q_m$. Hence $\omega_{\Gamma'} = 0$ and $\omega_\Gamma = 0$. \square

4.3 Proof of the formality theorem

The proof will follow [C2]. Before starting the proof, let's recall the following well known fact of sheaf theory : If \mathcal{C}_1^* and \mathcal{C}_2^* are complexes of c-soft

sheaves and if Θ is a quasi-isomorphism from \mathcal{C}_1^* to \mathcal{C}_2^* , then $\Gamma(\Theta)$ is a quasi-isomorphism from $\Gamma(\mathcal{C}_1^*)$ to $\Gamma(\mathcal{C}_2^*)$.

We will adopt the following notations :

$\lambda_T^M : {}^E T_{poly}^*(M) \rightarrow {}^E \Omega(\mathcal{T}(M))$ is the inverse of the map \mathcal{H} .

$\lambda_D^M : {}^E D_{poly}^*(M) \rightarrow {}^E \Omega(\mathcal{D}(M))$ is the map μ'_M .

We set $\lambda_D^{\mathcal{O}^X} = \lambda_D$ and $\lambda_T^{\mathcal{O}^X} = \lambda_T$. From Kontsevitch's work (theorem 4.2.1), we know that there exists a fiberwise quasi-isomorphism of L_∞ -algebras \mathcal{U} from ${}^E \Omega(\mathcal{T})$ to ${}^E \Omega(\mathcal{D})$ whose Taylor coefficients will be denoted $\mathcal{U}^{[n]} : S^n({}^E \Omega(\mathcal{T})[1]) \rightarrow {}^E \Omega(\mathcal{D})$ (first we construct \mathcal{U} on an open subset trivializing E and then glue the L_∞ -morphisms). Using the explicit expression of $\mathcal{U}^{[n]}$ ([Ko], [AMM]), one sees easily that $\mathcal{U}^{[n]}$ still make sense if we replace the last argument by an element of ${}^E \Omega(\mathcal{T}(M))$. Thus we define $\mathcal{V}^{[n]} : S^n({}^E \Omega(\mathcal{T})[1]) \otimes {}^E \Omega(\mathcal{T}(M)) \rightarrow {}^E \Omega(\mathcal{D}(M))$ by

$$\forall \gamma_1, \dots, \gamma_n \in {}^E \Omega(\mathcal{T})[1], \forall \nu \in {}^E \Omega(\mathcal{T}(M)), \quad \mathcal{V}^{[n]}(\gamma_1, \dots, \gamma_n, \nu) = \mathcal{U}^{[n+1]}(\gamma_1, \dots, \gamma_n, \nu).$$

Thus we get the following diagram

$$\begin{array}{ccc} ({}^E \Omega(\mathcal{T}), 0, [,]_S) & \xrightarrow{\mathcal{U}} & ({}^E \Omega(\mathcal{D}), \partial, [,]_G) \\ \cdot_S \downarrow L_\infty\text{-mod} & & \cdot_G \downarrow L_\infty\text{-mod} \\ ({}^E \Omega(\mathcal{T}(M)), 0, \cdot_S) & \xrightarrow{\mathcal{V}} & ({}^E \Omega(\mathcal{D}(M)), \partial_M, \cdot_G) \end{array}$$

Let V be an open subset on which $E|_V$ is trivial. The differential ${}^E d$ (respectively ${}^E d_M$) is defined on ${}^E \Omega(\mathcal{T})|_V$ and ${}^E \Omega(\mathcal{D})|_V$ (respectively ${}^E \Omega(\mathcal{T}(M))|_V$ and ${}^E \Omega(\mathcal{D}(M))|_V$). As the quasi-isomorphisms of the previous diagram are fiberwise, we can add the differentials ${}^E d$ and ${}^E d_M$, in the previous quasi-isomorphism. We get a morphism of L_∞ -algebras

$$\overline{\mathcal{U}} : ({}^E \Omega(\mathcal{T})|_V, {}^E d, [,]_S) \rightarrow ({}^E \Omega(\mathcal{D})|_V, {}^E d + \partial, [,]_G)$$

and a morphism of L_∞ -modules over ${}^E \Omega(\mathcal{T})|_V$

$$\overline{\mathcal{V}} : ({}^E \Omega(\mathcal{T}(M))|_V, {}^E d_M, \cdot_S) \rightarrow ({}^E \Omega(\mathcal{D}(M))|_V, {}^E d_M + \partial_M, \cdot_G)$$

We endow $\mathcal{B} = \mathcal{T}(M)|_V$ or $\mathcal{D}(M)|_V$ with the filtration

$$F^p({}^E \Omega(\mathcal{B})) = \bigoplus_{k \geq p} {}^E \Omega^k(\mathcal{B})$$

A spectral sequence argument shows that $\overline{\mathcal{U}}$ and $\overline{\mathcal{V}}$ are quasi-isomorphisms (see [C2] and [CDH] for details). Thus, we have the following diagram where the horizontal arrows are quasi-isomorphisms

$$\begin{array}{ccc} \left({}^E\Omega(\mathcal{T})|_V, {}^E d, [,]_S \right) & \xrightarrow{\overline{\mathcal{U}}} & \left({}^E\Omega(\mathcal{D})|_V, {}^E d + \partial, [,]_G \right) \\ \cdot_S \downarrow L_\infty\text{-mod} & & \cdot_G \downarrow L_\infty\text{-mod} \\ \left({}^E\Omega(\mathcal{T}(M))|_V, {}^E d_M, \cdot_S \right) & \xrightarrow{\overline{\mathcal{V}}} & \left({}^E\Omega(\mathcal{D}(M))|_V, {}^E d_M + \partial_M, \cdot_G \right) \end{array}$$

On V , the Fedosov differential can be written $D_M = {}^E d_M + B$ with

$$B = \sum_{p=0}^{\infty} \xi^i B_{i,j_1,\dots,j_p}(x) y^{j_1} \dots y^{j_p} \frac{\partial}{\partial y^k}.$$

We set $D = D_{\mathcal{O}_X}$. The element B of ${}^E\Omega^1(\mathcal{T}^0)|_V$ is a Maurer Cartan element of the (filtered) sheaf of DGLAs $\left({}^E\Omega(\mathcal{T})|_V, {}^E d, [,]_S \right)$. This means that $\left({}^E\Omega(\mathcal{T}(M))|_V, D_M, \cdot_S \right)$ is obtained from $\left({}^E\Omega(\mathcal{T}(M))|_V, {}^E d_M, \cdot_S \right)$ via the twisting procedure by the Maurer Cartan element B ([D2]). We know that $\sum_{n \geq 1} \frac{\mathcal{U}^{[n]}(B^n)}{n!}$ is a Maurer Cartan section of $\left({}^E\Omega(\mathcal{D})|_V, {}^E d + \partial, \cdot_G \right)$. But, due

to property (3) of U , $\sum_{n \geq 1} \frac{\mathcal{U}^{[n]}(B^n)}{n!} = B$. Twisting $\overline{\mathcal{U}}$ and $\overline{\mathcal{V}}$ by the Maurer Cartan element B ([D2]), we get the following diagram where the horizontal arrows are quasi-isomorphism

$$\begin{array}{ccc} \left({}^E\Omega(\mathcal{T})|_V, D, [,]_S \right) & \xrightarrow{\overline{\mathcal{U}}^B} & \left({}^E\Omega(\mathcal{D})|_V, D + \partial, [,]_G \right) \\ \cdot_S \downarrow L_\infty\text{-mod} & & \cdot_G \downarrow L_\infty\text{-mod} \\ \left({}^E\Omega(\mathcal{T}(M))|_V, D_M, \cdot_S \right) & \xrightarrow{\overline{\mathcal{V}}^B} & \left({}^E\Omega(\mathcal{D}(M))|_V, D_M + \partial_M, \cdot_G \right) \end{array}$$

$\overline{\mathcal{U}}^B$ and $\overline{\mathcal{V}}^B$ do not depend on the choice of the trivialization of $E|_V$ and hence are a well defined morphisms of L_∞ -algebras and L_∞ -modules respectively. Indeed the only term in B that depends on the coordinates is $\Gamma = -\xi^i \Gamma_{i,j}^k y^j \frac{\partial}{\partial y^k}$ and it is linear in the fiber coordinates y^i so that it

does neither contribute to $\overline{\mathcal{U}}^B$ nor to $\overline{\mathcal{V}}^B$ thanks to property (4) of U (see [D1], [C1], [D2], [CDH] for details). Hence $\overline{\mathcal{U}}^B$ and $\overline{\mathcal{V}}^B$ are defined globally and we get the following diagram.

$$\begin{array}{ccc} \left({}^E\Omega(\mathcal{T}), D, [\cdot, \cdot]_S \right) & \xrightarrow{\overline{\mathcal{U}}^B} & \left({}^E\Omega(\mathcal{D}), D + \partial, [\cdot, \cdot]_G \right) \\ \cdot_S \downarrow L_\infty\text{-mod} & & \cdot_G \downarrow L_\infty\text{-mod} \\ \left({}^E\Omega(\mathcal{T}(M)), D_M, \cdot_S \right) & \xrightarrow{\overline{\mathcal{V}}^B} & \left({}^E\Omega(\mathcal{D}(M)), D_M + \partial_M, \cdot_G \right) \end{array}$$

The following lemma shows that the map $\lambda_D^M(X)$ (and hence $\lambda_D(X)$) is a quasi-isomorphism from $\left[\Gamma \left({}^E D_{poly}(M) \right), \partial_M \right]$ to $\left[\Gamma \left({}^E\Omega(\mathcal{D}(M)) \right), D_M + \partial_M \right]$.

Lemma 4.3.1 *The natural inclusion $\iota : \left[\Gamma(\mathcal{D}^*(M) \cap \text{Ker} D_M), \partial_M \right] \hookrightarrow \left[\Gamma(\Omega^*(\mathcal{D}(M))), D_M + \partial_M \right]$ is a quasi-isomorphism.*

Proof of the lemma :

Consider a decomposition of $\text{Ker}(D_M + \partial_M)$ of the form

$$Y \oplus \text{Im}(D_M + \partial_M) = \text{Ker}(D_M + \partial_M).$$

One may construct a map $V : \text{Ker}(D_M + \partial_M) \rightarrow \Gamma(\Omega(\mathcal{D}(M)))$ such that

- i) for any x in $\text{Ker}(D_M + \partial_M)$, $x - (D_M + \partial_M)(V(x)) \in \Gamma(\mathcal{D}(M) \cap \text{Ker} D_M)$
- ii) If $x \in \text{Im}(D_M + \partial_M)$, $V(x)$ is a preimage of x by $D_M + \partial_M$.

It is enough to construct $V(x)$ for x in Y . Write $x = x_r + \dots + x_0$ with $x_i \in \Gamma(\Omega^i(\mathcal{D}(M)))$. The equality $(D_M + \partial_M)(x) = 0$ implies $D_M(x_r) = 0$ (because ∂_M preserves the exterior degree). Then using the exactness of D_M , we construct a map $V_r : Y \rightarrow \Gamma(\Omega^{\leq r-1}(\mathcal{D}(M)))$ such that for any x in Y , $x - (D_M + \partial_M)V_r(x)$ has maximal exterior degree inferior or equal to $r - 1$. Going on like this, we construct V .

We may now exhibit an inverse to $H^i(\iota)$. With obvious notations, we have

$$H^i(\iota)^{-1}([\mu]) = [\mu - (D_M + \partial_M)V(\mu)].$$

This finishes the proof of the lemma. \square

As $\lambda_D^M(X)$ is a quasi-isomorphism of L_∞ -modules over $\Gamma \left({}^E D_{poly}^* \right)$, there exists a quasi-isomorphism of L_∞ -modules over $\Gamma \left({}^E D_{poly}^* \right)$, $\alpha_D^M : \left[\Gamma \left({}^E\Omega(\mathcal{D}(M)) \right), D_M + \partial_M \right] \rightarrow \left[\Gamma \left({}^E D_{poly}^*(M) \right), \partial_M \right]$ such that

$H^i(\alpha_D^{M[1]}) = H^i(\lambda_D^M)^{-1}$ (see [AMM] for the case of L_∞ algebras). The morphism $\mathcal{V}_M = \alpha_D^M \circ \overline{\mathcal{V}}^B(X) \circ \lambda_T^M(X)$ is a quasi-isomorphism of L_∞ -modules over $\Gamma(E T_{poly}^*)$ from $\Gamma(E T_{poly}^*(M))$ to $\Gamma(E D_{poly}^*(M))$. One checks easily that $\mathcal{V}_M^{[0]}$ induces U_{HKR}^M in cohomology.

Inverting λ_D into a quasi-isomorphism of L_∞ algebras provides Calaque's quasi-isomorphism of L_∞ algebras Υ from $\Gamma(E T_{poly}^*)$ to $\Gamma(E D_{poly}^*)$ ([C2]). This finishes the proof of the theorem 3.4.1. \square

4.4 Local expression of \mathcal{V}_M in the case of the tangent bundle of \mathbb{R}^d .

In this section, assume that $X = \mathbb{R}^d$ and $E = T\mathbb{R}^d$. We choose the connection whose Christoffel symbols are 0. Thus, we have

$$\nabla(f \frac{\partial}{\partial x^i}) = df \frac{\partial}{\partial x^i}.$$

In this case $A = 0$ and $D = d_E - \delta$. If u is in $E T_{poly}(M)$ or $E D_{poly}(M)$, a computation shows that

$$\lambda(u) = \sum_{\alpha_1, \dots, \alpha_d} \frac{(y^1)^{\alpha_1}}{\alpha_1!} \dots \frac{(y^d)^{\alpha_d}}{\alpha_d!} \left[\left(\frac{\partial}{\partial x^1} \right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x^d} \right)^{\alpha_d} \right] \cdot u.$$

For example

$$\begin{aligned} & \lambda_T \left(\gamma^{j_1, \dots, j_p} \frac{\partial}{\partial x^{j_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{j_p}} \right) \\ &= \sum_{\alpha_1, \dots, \alpha_d} \frac{(y^1)^{\alpha_1}}{\alpha_1!} \dots \frac{(y^d)^{\alpha_d}}{\alpha_d!} \left[\left(\frac{\partial}{\partial x^1} \right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x^d} \right)^{\alpha_d} (\gamma^{j_1, \dots, j_p}) \right] \frac{\partial}{\partial y^{j_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{j_p}} \\ &= \lambda_T(\gamma^{j_1, \dots, j_p}) \frac{\partial}{\partial y^{j_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{j_p}}. \end{aligned}$$

From lemma 4.2.3, we see that $\overline{\mathcal{V}}^B = \overline{\mathcal{V}}$. If a is in $\mathcal{O}_{\mathbb{R}^d}$, one has

$$\frac{\partial}{\partial y^i} \lambda(a) = \lambda \left(\frac{\partial a}{\partial x^i} \right).$$

Then it is easy to see that in this special case $\overline{\mathcal{V}} \circ \lambda_T$ takes its values in $D_{poly} \cap \text{Ker} D$.

B_Γ makes sense if we change the last argument by a polydifferential operator with coefficients in M and it is not hard to see that

$$\mathcal{V}_M^{[n]} = \sum_{\Gamma \in G_{n+1,m}} W_\Gamma B_\Gamma.$$

5 Applications

In this section, we set $O = \Gamma(\mathcal{O}_X)$. Let E be a Lie algebroid, \mathcal{M} a $D(E)$ -module and $M = \Gamma(\mathcal{M})$. We denote by $\mathcal{V}_\mathcal{M}$ the quasi-isomorphism of L_∞ -modules over $\Gamma(E T_{poly}^*)[[h]]$ from $\Gamma(E T_{poly}^*(\mathcal{M}))[[h]]$ to $\Gamma(E D_{poly}^*(\mathcal{M}))[[h]]$ given by theorem 3.4.1. Then $\mathcal{V}_{\mathcal{O}_X} = \Upsilon$ is the L_∞ -quasi-isomorphism of DGLAs from $\Gamma(E T_{poly}^*)[[h]]$ to $\Gamma(E D_{poly}^*)[[h]]$ constructed by D. Calaque ([C1]). Let π_h be a Maurer Cartan element of $\Gamma(E T_{poly}^*)[[h]]$. This means that

$$\pi_h \in \Gamma(E T_{poly}^1) [[h]] \quad \text{and} \quad [\pi_h, \pi_h]_S = 0.$$

Then it is well known that $\sum_{n \geq 1} \frac{1}{n!} \Upsilon^{[n]}(\pi_h, \dots, \pi_h)$ is a Maurer Cartan element of $\Gamma(E D_{poly}^*)[[h]]$ (see [AMM] p. 63). We set

$$\Pi_h = 1 \otimes 1 + \sum_{n \geq 1} \frac{1}{n!} \Upsilon^{[n]}(\pi_h, \dots, \pi_h).$$

As $\Gamma(E T_{poly}^*(\mathcal{M}))[[h]]$ is a module over the DGLA $\Gamma(E T_{poly}^*)[[h]]$, the map

$$\begin{aligned} \pi_h \cdot - : \Gamma(E T_{poly}^k(\mathcal{M}))[[h]] &\rightarrow \Gamma(E T_{poly}^{k+1}(\mathcal{M}))[[h]] \\ y &\mapsto \pi_h \cdot_S y \end{aligned}$$

is a differential over $\Gamma(E T_{poly}^*(\mathcal{M}))[[h]]$ (see [D2] proposition 3 of section 2.3). Similarly, $\Pi_h \cdot_G -$ defines a differential on $\Gamma(E D_{poly}^*(\mathcal{M}))[[h]]$.

Proposition 5.0.1 *The map*

$$\begin{aligned} (\mathcal{V}_\mathcal{M})'_\pi : \left(\Gamma(E T_{poly}^*(\mathcal{M}))[[h]], \pi_h \cdot_S - \right) &\rightarrow \left(\Gamma(E D_{poly}^*(\mathcal{M}))[[h]], \Pi_h \cdot_G - \right) \\ y &\mapsto \sum_{p=0}^{\infty} \frac{1}{p!} \mathcal{V}_\mathcal{M}^{[p]}(\pi_h, \dots, \pi_h, y) \end{aligned}$$

is a quasi-isomorphism.

Proof of the proposition :

The proposition follows from proposition 3 of paragraph 2.3 of [D2] and the definition of the L_∞ -module structure of $\Gamma \left({}^E D_{poly}^*(\mathcal{M}) \right)$ over $\Gamma \left({}^E T_{poly}^* \right)$. \square

If E is a Lie algebroid equipped with an E -bivector $\pi \in \Gamma(\wedge^2 E)$ satisfying $[\pi, \pi] = 0$, it is called a Poisson Lie algebroid. If $E = TX$, we recover Poisson manifolds. Then, one can construct a Lie algebroid structure on E^* in the following way. Let π^\sharp be the bundle map from E^* to E associated to π and $\omega_* = \omega \circ \pi^\sharp : E^* \rightarrow TX$. Define a Lie bracket on E^* by :

$$\forall \theta, \eta \in E^*, \quad [\theta, \eta] = L_{\pi^\sharp \theta}(\eta) - L_{\pi^\sharp \eta}(\theta) - d\pi(\theta, \eta)$$

where L denotes the Lie derivative. Then E^* , endowed with the bracket above and the anchor ω_* is a Lie algebroid ([KM], [MX]) and E is a Lie bialgebroid. The differential of the Lie cohomology complex of E^* is $d_* = [\pi, -] : \Gamma(\wedge^k E) \rightarrow \Gamma(\wedge^{k+1} E)$.

Assume that we are in the case where E is a Poisson Lie algebroid with Poisson bivector π . Then, in the proposition above one may take $\pi_h = h\pi$ and Calaque ([C1]) shows that Π_h is a twistor for the standard Hopf algebroid $U(\Gamma(E))[[h]]$ ([X]).

From now on, we assume that $E = TX$ and that π is a Poisson bracket on X . Then the twistor Π_h defines a star product on $O[[h]]$ ([X]) in the following way

$$\forall (f, g) \in O, \quad \Pi_h(f, g) = f *_h g.$$

Set

$$f *_h g = fg + \sum_{i=1}^{\infty} B_i(f, g) h^i$$

Proposition 5.0.2 *$M[[h]]$ can be endowed with an $O[[h]] \otimes O[[h]]^{op}$ -module structure as follows : for all a in O and all m in M ,*

$$a * m = a \cdot m + \sum_{i=1}^{\infty} h^i B_i(a, -) \cdot m, \quad m * a = a \cdot m + \sum_{i=1}^{\infty} h^i B_i(-, a) \cdot m$$

Proof of the proposition :

The proof of the proposition is a straightforward verification using the associativity of the star product. \square

Applying the exact functor $N \mapsto N[[h]]$, we get an injection

$$\Gamma \left({}^E D_{poly}^k(\mathcal{M}) \right) [[h]] \hookrightarrow Hom_{\mathbb{R}[[h]]} \left(O[[h]]^{\otimes_{\mathbb{R}[[h]]}^{k+1}}, M[[h]] \right).$$

The image of $\Gamma \left({}^E D_{poly}^*(\mathcal{M}) \right) [[h]]$ in $Hom_{\mathbb{R}[[h]]} \left(O[[h]]^{\otimes_{\mathbb{R}[[h]]}^{*+1}}, M[[h]] \right)$ will be denoted $Homdif_{\mathbb{R}[[h]]} \left(O[[h]]^{\otimes_{\mathbb{R}[[h]]}^{*+1}}, M[[h]] \right)$.

Recall that the Hochschild cohomology of $O[[h]]$ with values in the bi-module $M[[h]]$, $HH^*(O[[h]], M[[h]])$, is the cohomology of the complex $(Hom_{\mathbb{R}[[h]]}(O[[h]]^{\otimes_{\mathbb{R}[[h]]}^*}, M[[h]]), \beta)$ where, with obvious notations,

$$\begin{aligned} \beta(\lambda)(a_1, \dots, a_{n+1}) &= a_1 * \lambda(a_2, \dots, a_{n+1}) \\ &+ \sum_{0 < i < n+1} (-1)^i \lambda(a_1, \dots, a_i * a_{i+1}, \dots, a_{n+1}) \\ &+ (-1)^{n+1} \lambda(a_1, \dots, a_n) * a_{n+1}. \end{aligned}$$

Denote by $HH_{md}^*(O[[h]], M[[h]])$ the cohomology of the complex $(Homdif_{\mathbb{R}[[h]]}(O[[h]]^{\otimes^*}, M[[h]]), \beta)$.

The complex $(\Gamma({}^E T_{poly}^*(\mathcal{M}))[[h]], \pi_h \cdot S)$ computes the Lichnerowicz-Poisson cohomology of the $\mathbb{R}[[h]]$ -Poisson algebra (defined by the bivector π_h) $O[[h]]$ with coefficients in $M[[h]]$, $H_{Poisson}^i(O[[h]], M[[h]])$ ([Li], [Hu]). Denote by $H_{Poisson}^i(O[[h]], M[[h]])$ the Lichnerowicz-Poisson cohomology of the $\mathbb{R}[[h]]$ -Poisson algebra (defined by the bivector π_h) $O[[h]]$ with coefficients in $M[[h]]$ ([Hu]). It is computed by the complex $(\Gamma({}^E T_{poly}^*(\mathcal{M}))[[h]], \pi_h \cdot S)$. The complex $(\Gamma({}^E D_{poly}^*(\mathcal{M}))[[h]], \Pi_h \cdot G)$ computes $HH_{md}^*(O[[h]], M[[h]])$. We get the following corollary :

Corollary 5.0.3 *One has an isomorphism*

$$H_{Poisson}^i(O[[h]], M[[h]]) \simeq HH_{md}^i(O[[h]], M[[h]]).$$

The exterior product, which will be denoted by \wedge , endows $H^*(\Gamma({}^E T_{poly}^*), [\pi_h, \cdot])$ with an associative supercommutative algebra structure. It also endows $H^*(\Gamma({}^E T_{poly}^*(\mathcal{M})), \pi_h \cdot S)$ with a $[H^*(\Gamma({}^E T_{poly}^*), [\pi_h, \cdot]), \wedge]$ -module structure.

To simplify the notation, from now on, we write Π instead of Π_h . D_{poly}^* is endowed with an associative graded product, \sqcup_Π , compatible with the differential $[\Pi, \cdot]$ defined by :

$$\begin{aligned} \forall t_1 \in \Gamma(D_{poly}^{k_1-1}), \forall t_2 \in \Gamma(D_{poly}^{k_2-1}), \forall a_1, \dots, a_{k_1+k_2} \in O, \\ (t_1 \sqcup_\Pi t_2)(a_1, \dots, a_{k_1+k_2}) = t_1(a_1, \dots, a_{k_1}) \star_h t_2(a_{k_1+1}, \dots, a_{k_1+k_2}) \end{aligned}$$

Thus, $[H^*(\Gamma(D_{poly}), [\Pi, \cdot]), \sqcup_\Pi]$ is an associative graded algebra. $t_1 \sqcup_\Pi t_2$ is also defined if $t_2 \in \Gamma(D_{poly}^{k_2-1}(\mathcal{M}))$. Thus, $[H^*(\Gamma(D_{poly}(\mathcal{M})), \Pi \cdot_G), \sqcup_\Pi]$ is a $[H^*(\Gamma(D_{poly}), [\Pi, \cdot]), \sqcup_\Pi]$ -module.

If $X = \mathbb{R}^d$ and $E = T\mathbb{R}^d$, Kontsevich has proved ([Ko], see [MT] for a detailed proof) that the algebras $[H^*(\Gamma(T_{poly}^*), [\pi_h, \cdot]), \wedge]$ and $[H^*(\Gamma(D_{poly}), [\Pi, \cdot]), \sqcup_\Pi]$ are isomorphic. We will extend this result to our case.

Remark : In [CFT], a star product $*$ is constructed on any manifold X so that the algebras $[H^0(\Gamma(T_{poly}^*), [\pi_h, \cdot]), \wedge]$ and $[H^0(\Gamma(D_{poly}), [* , \cdot]), \sqcup_\Pi]$ are isomorphic.

Theorem 5.0.4 *Assume that $X = \mathbb{R}^d$ and $E = T\mathbb{R}^d$. The $[H^*(\Gamma(T_{poly}^*), [\pi_h, \cdot]), \wedge] \simeq [H^*(\Gamma(D_{poly}), [\Pi, \cdot]), \sqcup_\Pi]$ -modules $[H^*(\Gamma(T_{poly}^*(\mathcal{M})), \pi_h \cdot_S), \wedge]$ and $[H^*(\Gamma(D_{poly}(\mathcal{M})), \Pi \cdot_G), \sqcup_\Pi]$ are isomorphic.*

Proof of the theorem 5.0.4

In this proof, we keep the notations of the proof of the formality theorem (paragraph 4.3). We could reproduce the proof of [MT] using the explicit expression we found for $\mathcal{V}_\mathcal{M}$ in the paragraph 4.4. We will use the decomposition $\mathcal{V}_\mathcal{M} = \lambda_D^{-1} \circ \overline{\mathcal{V}} \circ \lambda_T$ and use the results of [MT]. Put

$$\overline{\Pi} = \sum_{n \geq 1} \mathcal{U}^{[n]}(\lambda_T(\pi_h), \dots, \lambda_T(\pi_h)).$$

Lemma 5.0.5 *Let k_1 and k_2 be in \mathbb{N} . If $\tau_1 \in \Gamma(\mathcal{T}_{poly}^{k_1-1})$, $\tau_2 \in \Gamma(\mathcal{T}_{poly}^{k_2-1}(\mathcal{M}))$ and $m = k_1 + k_2$ then one has*

$$\begin{aligned} & \overline{\mathcal{V}}'_{\lambda_T(\pi_h)}(\tau_1 \wedge \tau_2) - \overline{\mathcal{U}}'_{\lambda_T(\pi_h)}(\tau_1) \sqcup_{\overline{\Pi}} \overline{\mathcal{V}}'_{\lambda_T(\pi_h)}(\tau_2) = \\ & \sum_{n \geq 0} \frac{h^n}{n!} \sum_{\Delta \in G_{n+2, m-1}} a_{\Delta} \overline{\Pi} \cdot_G B_{\Delta}(\lambda_T(\pi) \otimes \dots \otimes \lambda_T(\pi) \otimes \tau_1 \otimes \tau_2) \\ & + \sum_{n \geq 0} \frac{h^n}{n!} \sum_{\Delta \in G_{n+1, m}} b_{\Delta} (-1)^{(k_1-1)k_2} B_{\Delta}(\lambda_T(\pi) \otimes \dots \otimes \lambda_T(\pi) \otimes [\lambda_T(\pi), \tau_1] \otimes \tau_2) \\ & \sum_{n \geq 0} \frac{h^n}{n!} \sum_{\Delta \in G_{n+1, m}} b_{\Delta} (-1)^{k_1(k_2-1)} B_{\Delta}(\lambda_T(\pi) \otimes \dots \otimes \lambda_T(\pi) \otimes \tau_1 \otimes \lambda_T(\pi) \cdot_S \tau_2) \end{aligned}$$

where a_{Δ} and b_{Δ} are real.

Proof of the lemma 5.0.5 :

The lemma 5.0.5 is proved for $\mathcal{M} = \mathcal{O}_X$ in [MT]. Actually, the formula of lemma 5.0.5 is slightly different from that of [MT]. To get it, one has to reproduce the proof of [MT] and make play to the vertices $n-1$ and n the role played by the vertices 1 and 2. Hence the lemma 5.0.5 holds for τ_2 in $\Gamma(\mathcal{T}_{poly}^{k_2-1}) \otimes_O M$. We will now prove that it is true for τ_2 in $\Gamma(\mathcal{T}_{poly}^{k_2-1}(\mathcal{M}))$. If we apply it to (f_1, \dots, f_m) in $\mathbb{R}[[y^1, \dots, y^d]]^m$, the relation of the lemma 5.0.5 can be written $\sum_{n \geq 0} h^n F_n = \sum_{n \geq 0} h^n G_n$ where the F_n 's and the G_n 's are maps from $\Gamma(\mathcal{T}_{poly}^{k_2-1}) \otimes_O M$ to $M[[y^1, \dots, y^d]]$. Let I be the ideal of $O[[y^1, \dots, y^d]]$ generated by y^1, \dots, y^d . The F_n 's and the G_n 's are continuous for the I -adic topology. This is a consequence of the following two remarks.

- Let $\gamma_1, \dots, \gamma_p$ be elements of $\Gamma(\mathcal{T}_{poly})$ and let (g_1, \dots, g_m) be elements of $O[[y^1, \dots, y^d]]$. Let Γ be an admissible graph of type $(p+1, m)$. The map

$$\begin{aligned} \Gamma(\mathcal{T}_{poly}^{k_2-1}) \otimes_O M & \rightarrow M[[y^1, \dots, y^d]] \\ \mu & \mapsto B_{\Gamma}(\gamma_1, \dots, \gamma_p, \mu)(g_1, \dots, g_m) \end{aligned}$$

is continuous for the I -adic topology as it sends $I^N \Gamma(\mathcal{T}_{poly}^{k_2-1}) \otimes_O M$ to $I^{N-p} M[[y^1, \dots, y^d]]$.

- Let Γ be an admissible graph of type $(p, 2)$ and let g be an element of $O[[y^1, \dots, y^d]]$. The map

$$\begin{aligned} O[[y^1, \dots, y^d]] \otimes_O M & \rightarrow M[[y^1, \dots, y^d]] \\ \mu & \mapsto B_{\Gamma}(\lambda_T(\pi), \dots, \lambda_T(\pi))(f, \mu) \end{aligned}$$

is continuous for the I -adic topology as it sends $I^N O[[y^1, \dots, y^d]] \otimes_O M$ to $I^{N-p} M[[y^1, \dots, y^d]]$.

This finishes the proof of the lemma 5.0.5. \square

Now, we go back to the proof of the theorem 5.0.4

Let t_1 be in $\Gamma(T_{poly}^{k_1-1})[[h]] \cap Ker[\pi_h,]$ and t_2 be in $\Gamma(T_{poly}^{k_2-1}(\mathcal{M}))[[h]] \cap Ker(\pi_h \cdot_S)$. We apply the lemma 5.0.5 to $\tau_1 = \lambda_T(t_1)$ and $\tau_2 = \lambda_T^{\mathcal{M}}(t_2)$. We get

$$\begin{aligned} & \overline{\mathcal{V}}'_{\lambda_T(\pi_h)}(\lambda_T(t_1) \wedge \lambda_T^{\mathcal{M}}(t_2)) - \overline{\mathcal{U}}'_{\lambda_T(\pi_h)}(\lambda_T(t_1)) \sqcup_{\overline{\Pi}} \overline{\mathcal{V}}'_{\lambda_T(\pi_h)}(\lambda_T^{\mathcal{M}}(t_2)) \\ &= \sum_{n \geq 0} \frac{h^n}{n!} \sum_{\Delta \in G_{n+2, m-1}} a_{\Delta} \overline{\Pi} \cdot_G B_{\Delta}(\lambda_T(\pi) \otimes \dots \otimes \lambda_T(\pi) \otimes \lambda_T(t_1) \otimes \lambda_T^{\mathcal{M}}(t_2)). \end{aligned}$$

Apply $(\lambda_D^{\mathcal{M}})^{-1}$ and use the following facts :

- $\lambda_D^{-1}(\overline{\Pi}) = \Pi$.
- With obvious notations, one has.

$$\lambda_D(\sigma_1) \sqcup_{\overline{\Pi}} \lambda_D^{\mathcal{M}}(\sigma_2) = \lambda_D^{\mathcal{M}}(\sigma_1 \sqcup_{\Pi} \sigma_2).$$

- $B_{\Delta}(\lambda_T(\pi), \dots, \lambda_T(\pi), \lambda_T(t_1), \lambda_T^{\mathcal{M}}(t_2)) = \lambda_D^{\mathcal{M}}(B_{\Delta}(\pi, \dots, \pi, t_1, t_2))$

We get

$$(\mathcal{V}_{\mathcal{M}})'_{\pi}(t_1 \wedge t_2) - \mathcal{U}'_{\pi}(t_1) \sqcup_{\Pi} (\mathcal{V}_{\mathcal{M}})'_{\pi}(t_2) = \sum_{n \geq 0} \frac{h^n}{n!} \sum_{\Delta \in G_{n+2, m-1}} a_{\Delta} \Pi \cdot_G B_{\Delta}(\pi \otimes \dots \otimes \pi \otimes t_1 \otimes t_2).$$

The right hand side is a coboundary for the Hochschild cohomology complex. This finishes the proof of the theorem 5.0.4. \square

Remark :

Assume that X is the dual of a real Lie algebra endowed with its Kirillov-Kostant-Souriau Poisson structure denoted by π . Recall that if ξ and η are elements of \mathfrak{g} considered as linear forms on \mathfrak{g}^* , then

$$\pi(\xi, \eta) = [\xi, \eta].$$

If $M = \mathcal{O}_X$, the isomorphism given by theorem 5.0.4 has been studied. If $i = 0$, it gives Duflo's isomorphism ([Du], [Ko]). By analyzing which graphs

contributes to $(\mathcal{V}_{\mathcal{M}})'_{\pi}$, Pevsner and Torossian ([PT]) have shown that that Duflo's isomorphism extends to an isomorphism from $H^*_{Poisson}(\mathfrak{g}, S(\mathfrak{g}))$ to $H^*(\mathfrak{g}, U(\mathfrak{g}))$.

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